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Exactness of conventional and supersymmetric JWKB formulae and their relation to the shape invariance property of potentials

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Abstract. A mechanism for the JWKB formulae in one-dimensional quantum mechanics to quantize energy levels exactly is discussed. This mechanism is most easily applied when considered potentials are represented (by a suitable change of variable) as periodic ones. To be successful, the mechanism demands potentials with no more than one second-order pole and no more than two (occasionally four) turning points in the basic period strips. It then selects 11 potentials which have been known in the literature for a long time. It is also shown that the exactness of the supersymmetric (SUSY) JWKB formulae for these 11 potentials follows directly from the corresponding exactness of the conventional ones being a consequence of the singularity structures of the potentials. A relation of these singularity structures to the shape invariance symmetry of the quantized potentials is shown to guarantee the conventional and SUSY JWKB formulae to be exact simultaneously. The relevant two non-perturbative theorems describing these facts are formulated and proved.

1. Introduction

It has long been known that for some number of potentials in one-dimensional (1D) quantization problems (and also in the cases of n-dimensional problems which can be reduced to 1D ones) their corresponding JWKB-quantization formulae for energy levels (or some of their generalizations [1] or modifications [6]) are exact [2–5, 11], whilst in most solvable cases (i.e. the ones for which their corresponding energy spectrum is known by other means) the JWKB quantization appears to be only approximate.

It was also noticed that the same solvable potentials which are quantized accurately by JWKB formulae are also quantized exactly when the supersymmetric (SUSY) modification of the JWKB method is used [7–9].

A class of shape invariant potentials are also known to provide examples of the *exact* SUSY JWKB (SJWKB) quantization formulae [28–31]. But, in general, the SJWKB formulae do not provide accurate quantization conditions in most cases of solvable potentials [10, 35, 36].

Therefore, the following questions still need to be answered:

(1) When can the JWKB formulae, both conventional and SUSY, be exact?
(2) Why are the conventional and SJWKB formulae exact simultaneously?
(3) Are there some (possibly simple) criteria which allow us to judge if a given JWKB (SJWKB) formula is exact?

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Before reporting our answers to the above questions let us briefly summarize the basic facts related directly to these questions and established by different authors.

First, Rosenzweig and Krieger [4] in the case of conventional JWKB formulae and Crescimanno [11] in the case of SUSY formulae have constructed the proofs of their exactness for some potentials based on the Fröman and Fröman [2] approach to 1D semiclassics. However, from our point of view, the correctness of these proofs seems to be doubtful because of erroneous calculations of necessary phases when functions have infinitely many roots and poles.

Secondly, Raghunathan *et al* [33] have shown that some SJWKB formulae are exact at the semiclassical level, whilst Dutt *et al* [30] have proved this for a class of shape invariant potentials.

Finally, Barclay and Maxwell [31] have used the result of Dutt *et al* [30] to suggest the form of superpotentials satisfying the shape invariance symmetry condition whilst Barclay *et al* [34] have shown that for another class of shape invariant potentials discovered by this group [35, 36] the corresponding SJWKB formulae are not exact.

The following results established in this paper form our answers to the questions posed above. In their formulation we have taken into account the fact that each 1D quantization problem can *always* be transformed into its periodic form (see section 2).

(a) A unifying condition for potentials to be quantized exactly by the JWKB (SJWKB) formulae is the shape invariance of these potentials accompanied by some particular analytical properties of the latter when considered as functions of *complex* superpotentials. If these properties are confirmed then both types of JWKB quantizations are exact simultaneously.

(b) The properties of the potentials mentioned in (a) are in direct relation to analytic properties and symmetries of the potentials considered on the complex plane of the *position* variable. The potentials to be provided with the properties mentioned above should have no more than two (occasionally four) turning points and no more than one pole (of the order of no higher than two) in their basic period strip.

The properties of (b) can be established independently of (a) and follow from a basic mechanism causing the JWKB formulae to be exact. An effective action of this mechanism depends on symmetry properties of the periodic Stokes graphs corresponding to the considered potentials.

Our analysis which made use of the mechanism can be therefore considered as *a method* of searching for solutions to the shape invariance condition when the latter is also expected to ensure that the JWKB (SJWKB) quantization is exact.

To obtain the results reported in (a) and (b) above, the fundamental solutions [1, 13–15, 17, 32] have been found to be the most appropriate. In this paper, the descriptions approximate semiclassical solutions, and JWKB approximations are exclusively understood as the corresponding approximations to the fundamental solutions. This choice has serious consequences for the form of the JWKB approximations which can differ substantially from the conventional ones [12, 24, 25]. In particular, the presence of simple and second-order poles in considered potentials generates unavoidable changes in the corresponding JWKB formulae.

Each set of the fundamental solutions is accompanied by the so-called Stokes graph. Both the fundamental solutions and the Stokes graphs provide us with a uniform and systematic way of solving any interesting 1D problem both exactly and in the semiclassical limit [1, 13–16]. The main property of the Stokes graph is to take into account *global* features of a given problem considered in the complex planes of variables entered into the problem (i.e. a position variable, energy, the Planck constant, some potential parameter(s), etc). It is just these global features

determining global structures of corresponding Stokes graphs which allow us to justify all the known cases of exact JWKB formulae, as well as to gain insight as to what decides whether a given JWKB formula can be exact or not.

This paper is organized in the following way.

In the next section, some necessary details of material from our earlier papers [13,14,16,23] are included to make the present paper selfcontained. In this section, a mechanism generating exact JWKB formulae is formulated and the periodicity of potentials and their Stokes graphs are established as the necessary conditions for the mechanism to work. However, since every 1D quantum mechanical problem can be formulated as a periodic one this condition rather suggests the best method by which the action of the mechanism can be studied. In consequence, the real conditions determining the possible success of the JWKB formula in being exact are related to the singularity structure and the distribution of turning points in the basic period strip, as was reported in point (b) above.

The conditions found in section 2 are next applied in section 3 to select ten periodic potentials with the exact JWKB formulae.

In section 4, the JWKB exactness of energy levels of some aperiodic potentials corresponding to the radial parts of the Coulomb and 3D isotropic harmonic potentials are shown to explicitly follow from the periodic ones as a result of the change-of-variable procedure which preserves the form of the Schrödinger equation [27]. In this section, the harmonic oscilator potential is also mentioned with its exceptional mechanism for JWKB formula exactness.

In section 5, a generalization of the results of the previous section is described invoking some of our earlier results.

In section 6, the SUSY version of the conventional exact JWKB formulae found in the preceding section is shown to be exact too, by direct calculations. The sufficient conditions for the simultaneous exactness of the JWKB and SJWKB formulae are established, one of which is the shape invariance of the considered potentials. Two corresponding theorems are formulated in this section.

In section 7, the results of this paper are summarized and some conclusions are drawn.

2. Global symmetries of Stokes graphs and quantization

2.1. Quantization

Consider the Schrödinger equation written in the following form:

$$\Psi''(x, E, \lambda) - \lambda^2 q(x, E, \lambda)\Psi(x, E, \lambda) = 0 \tag{2.1}$$

where: $\lambda^2 = 2m\hbar^{-2}$, $q(x, E, \lambda) = V(x, \lambda) - E$ and a potential $V(x, \lambda)$ is assumed to be a meromorphic function of x and λ with the following asymptotic behaviour for $\lambda \to +\infty$ ($\hbar \to 0$):

$$V(x, \lambda) \sim V_0(x) + \frac{1}{\lambda}V_1(x) + \frac{1}{\lambda^2}V_2(x) + \cdots. \tag{2.2}$$

Together with $q(x, E, \lambda)$, we shall consider a function $\tilde{q}(x, E, \lambda) \equiv q(x, E, \lambda) + \delta(x, E, \lambda)/\lambda^2$, where $\delta(x, E, \lambda)$ behaves according to (2.2) when $\lambda \to +\infty$. The necessity of introducing this term while constructing the fundamental solutions to (2.1) has been discussed in our recent paper [23]. The precise form of $\delta(x, E, \lambda)$ depends on the types of singularities of $q(x, E, \lambda)$ and, in particular, on whether the latter possesses simple or second-order poles (see below).

Let E be real and let $x_1, x_2, \ldots$, be the roots of $\tilde{q}(x, E, \lambda)$, $y_1, y_2, \ldots$ its simple poles and $z_k, k = 1, 2, \ldots$ its second-or higher-order poles. Some of them can therefore be real but the rest are complex and conjugated pairwise.

For each point $x_i, y_i, i = 1, 2, \ldots$, let us construct the actions

$$W_i^r(x, E, \lambda) = \int_{x_i}^{x} \sqrt{q(y, E, \lambda)}\, \mathrm{d}y$$
$$W_i^p(x, E, \lambda) = \int_{y_i}^{x} \sqrt{q(y, E, \lambda)}\, \mathrm{d}y. \tag{2.3}$$

A set of fundamental solutions is attached in a unique way to the so-called Stokes graph corresponding to a given potential $V(x)$. Each Stokes graph is a collection of lines (Stokes lines) in the complex x-plane which are loci of points where the real parts of action functions defined by (2.3) vanish. The fundamental solutions are defined in connected domains called sectors. Each sector contains one of the singular points z_i and its boundary consists of Stokes lines, x_i and the chosen z_i itself: see figure 1. Quantization of 1D quantum systems with the help of fundamental solutions and the construction of these solutions have been described in many of our earlier papers [1, 13, 14, 16, 23, 32]. A typical scheme of such a quantization particularly useful for meromorphic potentials has been discussed in [23]. We shall also adopt this scheme here. Nevertheless, to make our paper selfcontained we shall remind the reader of the basic ingredients of the scheme: namely, we consider the case of two real turning points, x_1, x_2, the rest being complex and conjugated pairwise (we assume $\tilde{q}(x, E, \lambda)$ to be real). It is assumed also that our physical problem is limited to a segment, $z_1 \leqslant x \leqslant z_2$, at the ends of which the potential has poles. In particular, we can push any of $z_{1,2}$ (or both of them) to $\mp\infty$, respectively.

To construct a pattern of the corresponding quantization condition for energy E and to simultaneously handle the cases of second- and higher-order poles we assume z_1 to be the second-order pole and z_2 to be the higher ones.

It is also necessary to fix, to some extent, the closest environment of the real axis of the x-plane in order to plot a piece of the Stokes graph which will give sufficient information to write the quantization condition. To this end, we assume x_3 and $\bar{x}_3$ as well as x_4 and $\bar{x}_4$ to be another four turning points, and z_3 and $\bar{z}_3$ another two second-order poles of $V(x, \lambda)$ closest to the real axis. Then a possible section of the Stokes graph is shown in figure 1 [23].

To this Stokes graph we can attach to each of its sectors, S_k, the corresponding fundamental solution, Ψ_k, having the following structure [2, 13, 14, 17, 23]:

$$\Psi_k(x) = \tilde{q}^{-\frac{1}{4}}(x) \mathrm{e}^{\sigma_k \lambda W_i(x)} \chi_k(x) \tag{2.4}$$

where

$$\chi_k(x) = 1 + \sum_{n \geqslant 1} \left[-\frac{\sigma_k}{2\lambda}\right]^n \int_{z_k}^{x} \mathrm{d}y_1 \int_{z_k}^{y_1} \mathrm{d}y_2$$
$$\ldots \int_{z_k}^{y_{n-1}} \mathrm{d}y_n \omega(y_1)\omega(y_2)\ldots\omega(y_n)(1 - \mathrm{e}^{-2\sigma_k\lambda(W_i(x) - W_i(y_1))})$$
$$\times (1 - \mathrm{e}^{-2\sigma_k\lambda(W_i(y_1) - W_i(y_2))})$$
$$\ldots (1 - \mathrm{e}^{-2\sigma_k\lambda(W_i(y_{n-1}) - W_i(y_n))}) \tag{2.5}$$

with

$$\omega(y) = \frac{\delta(y)}{\tilde{q}^{\frac{1}{2}}(y)} - \frac{1}{4}\frac{\tilde{q}''(y)}{\tilde{q}^{\frac{3}{2}}(y)} + \frac{5}{16}\frac{\tilde{q}'^2(y)}{\tilde{q}^{\frac{5}{2}}(y)}. \tag{2.6}$$

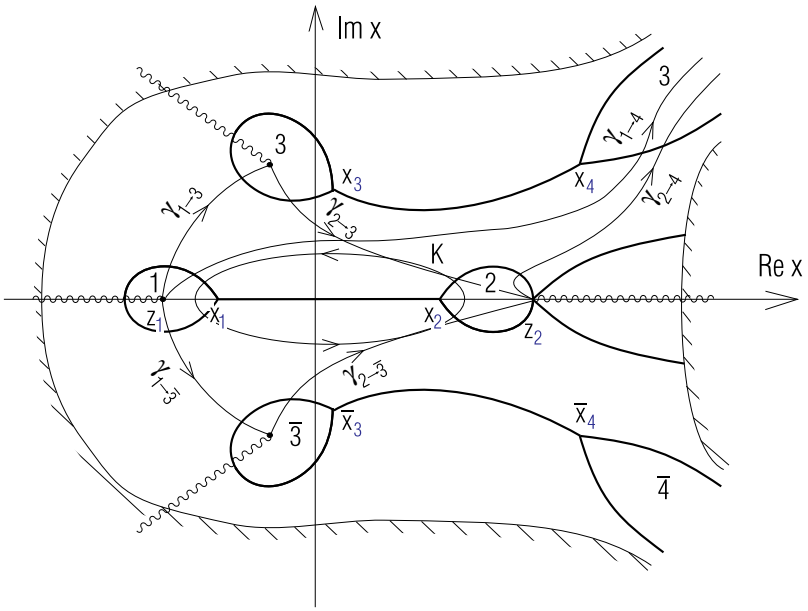

Figure 1. The Stokes graph corresponding to general quantization rule (2.7) [23].

In the above formulae x_i are some of the roots lying at the boundary of S_k and $\sigma_k = \pm 1$ is chosen each time so as to ensure a negative sign of Re $(\sigma_k W_i(x))$ for the whole sector S_k.

One of the conditions introduced earlier determining the function $\delta(x, E, \lambda)$ is to make all the multiple integrals in (2.5) convergent at their lower limits z_k. It appears that for this reason the function has to be defined as non-zero only if z_k is a second-order pole of the potential considered. The second reason appears when the fundamental solutions are to be continued to a point being a simple or double pole for the potential: namely, for both these cases we always have to correct the potential by the same term $\delta(x, E, \lambda) = (2(x - z_k))^{-2}$ at *each* simple or double pole of the potential. Of course, in the case of an infinite number of these singularities the arising infinite series has to be summed to some function having them as its simple and double poles. The δ-terms correcting the potentials considered in the above way we shall call Langer corrections [23] (see also [22] where this correction appeared for the first time in this role).

Despite the necessity to equip $\tilde{q}(x, E, \lambda)$ (and ω) in the δ-term, there is still a possibility to change the form of $\tilde{q}(x, E, \lambda)$ (and ω) by the substitution $\delta(x) \to \delta(x) + f(x, \lambda)$, where $f(x, \lambda)$ is an arbitrary meromorphic function of x not containing, however, the original singularities of $\tilde{q}(x, E, \lambda)$. By such a substitution both the original and the new solutions (2.4) coincide (up to a multiplicative constant, see [24] for details). We shall use this possibility in our further considerations.

There is no unique way of writing the quantization condition corresponding to the figure. Three possible forms of this condition can be written as [16, 23]:

$$\exp\left[-\lambda \oint_K \tilde{q}^{\frac{1}{2}}(x, \lambda, E)\,\mathrm{d}x\right] = -\frac{\chi_{1\to 3}(\lambda, E)\chi_{2\to\bar{3}}(\lambda, E)}{\chi_{1\to\bar{3}}(\lambda, E)\chi_{2\to 3}(\lambda, E)} = -\frac{\chi_{1\to 4}(\lambda, E)\chi_{2\to\bar{3}}(\lambda, E)}{\chi_{1\to\bar{3}}(\lambda, E)\chi_{2\to 4}(\lambda, E)} \tag{2.7}$$

and $\chi_{k\to j}(\lambda, E)k$, $j = 1, 2, 3, 4$ are calculated by (2.5) for $x \to z_j$. The closed integration path K is shown in figure 1. In the figure, the paths $\gamma_{1\to 3}$, $\gamma_{2\to 3}$, etc are the integration paths in formula (2.5), whilst the wavy lines designate corresponding cuts of the x-Riemann surface

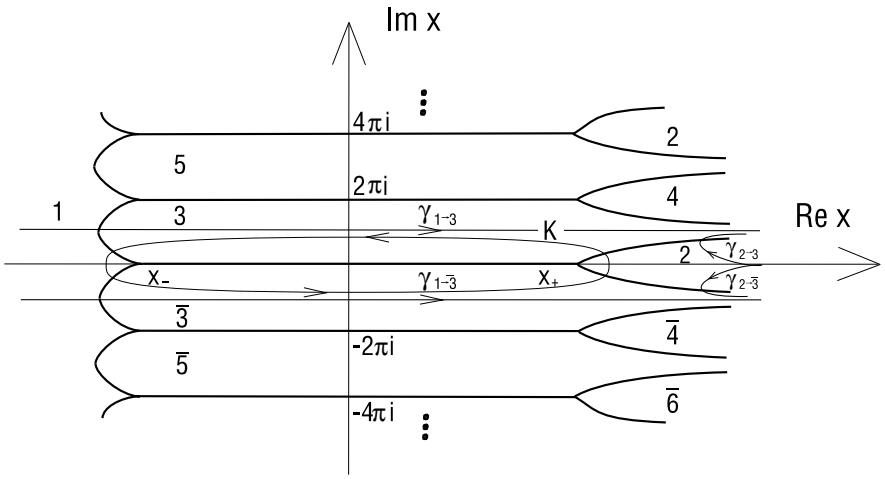

Figure 2. The Stokes graph for the Morse-type potential (3.2).

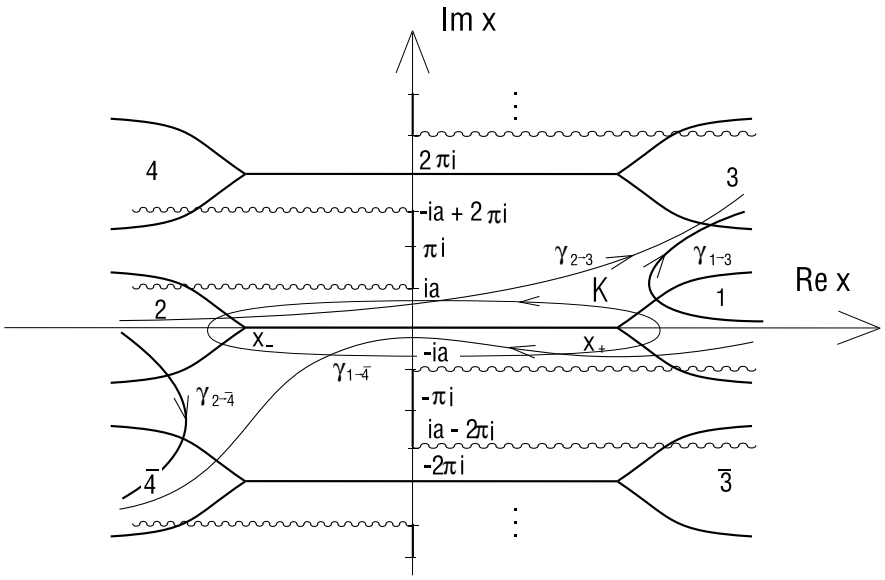

Figure 3. The Stokes graph for the first of the potentials (3.4).

upon which all the fundamental solutions are defined. The same designation conventions are maintained in the remaining figures 2–11.

2.2. Symmetry conditions

As we have already mentioned several times, each 1D quantum mechanical problem can be transformed into a periodic one by a suitable variable transformation: namely, if it is a non-periodic problem defined on a segment (z_1, z_2) (as in the case of figure 1) then the transformation $x \to (z_2 e^x + z_1)/(1 + e^x)$ can be taken. If it is defined on a half (z_1, ∞) of the real axis then the corresponding transformation is $x \to z_1 + e^x$. Finally, when it is defined on the whole real

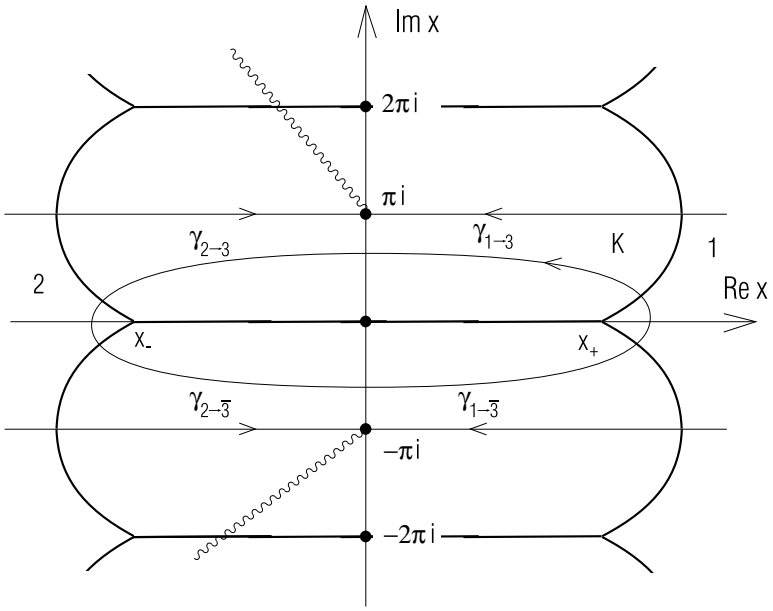

Figure 4. The Stokes graphs corresponding to the second of the potentials (3.4) and the quantization formulae (3.9).

axis, the transformation $x \to \sinh \frac{x}{2}$ can be performed.

We could proceed, therefore, considering only periodic potentials throughout. However, below we shall provide arguments to justify that the periodic formulation of the considered problems is the most suitable for the exactness problem discussed in this paper.

To this end, let us note that conditions (2.7) are *exact*. The lhs of the first condition has only the JWKB form. If we substitute each $\chi_{k\to j}(\lambda, E)$ in (2.7) by unity (which these coefficients approach when $\lambda \to +\infty$) we obtain the well known JWKB-quantization rule. But in this way the latter is, in general, only an approximation to (2.7). The exceptions to this are the following three cases:

(1^0**)** All $\chi_{k\to j}(\lambda, E)$ in (2.7) are really equal to (identical with) 1.
(2^0**)** They all cancel mutually for some reason.
(3^0**)** Both the above cases take place: i.e. some of $\chi_{k\to j}(\lambda, E)$ satisfy 1^0 and some 2^0.

The first case is very rare and the only known example of it is the harmonic oscillator potential [32]. This case needs, in fact, for a given $\chi_{k\to j}(\lambda, E)$ a possibility to deform its integration path properly to make all the integrations in (2.6) vanishing: i.e., this condition demands some particular topology of turning points on the x-plane to occur.

The next case, if it is not to happen accidentally, can take place due to the possibility of coefficients entering into formula (2.7) (where the coefficients can appear in pairs with their complex conjugate partners dividing them) or due to some possible symmetry of the potential $V(x, \lambda)$ relating to the χ-coefficients present in the formulae. We shall show in the following sections that the latter case is the main reason for all known cases of JWKB formulae which provide us with the exact quantization conditions. In fact, the symmetry properties of the potential, as well as a particular topology of its turning point and pole distribution cooperating

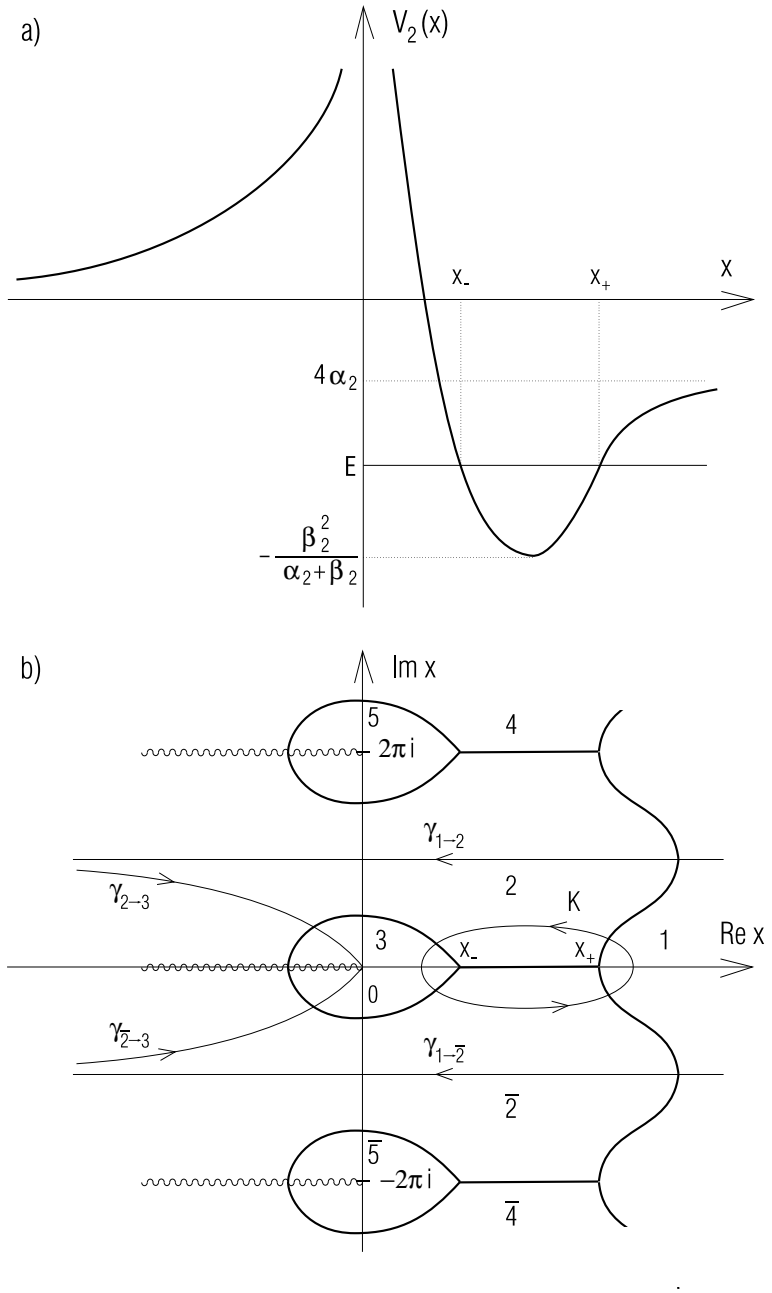

Figure 5. The second of potentials (3.10) and the Stokes graphs corresponding to the quantization formulae (3.12).

together, are the most frequent ways to realize JWKB formula exactness.

The above statement means that the corresponding Stokes graphs and the underlying potentials have to be *invariant* under the corresponding symmetry transformations. It is important to realize that this symmetry is just the *invariance*, the latter being the *necessary*

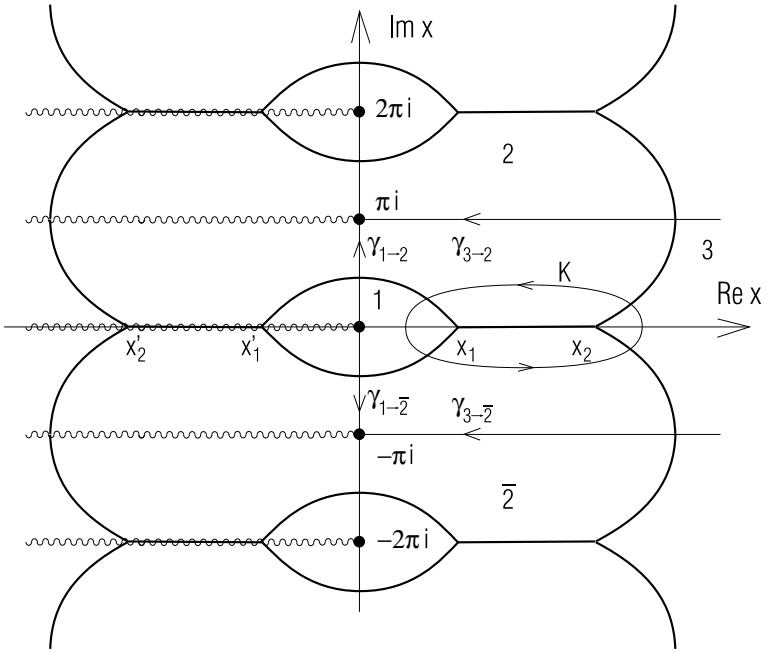

Figure 6. The Stokes graph corresponding to the formula (3.14) quantizing the potential of Pöschl and Teller.

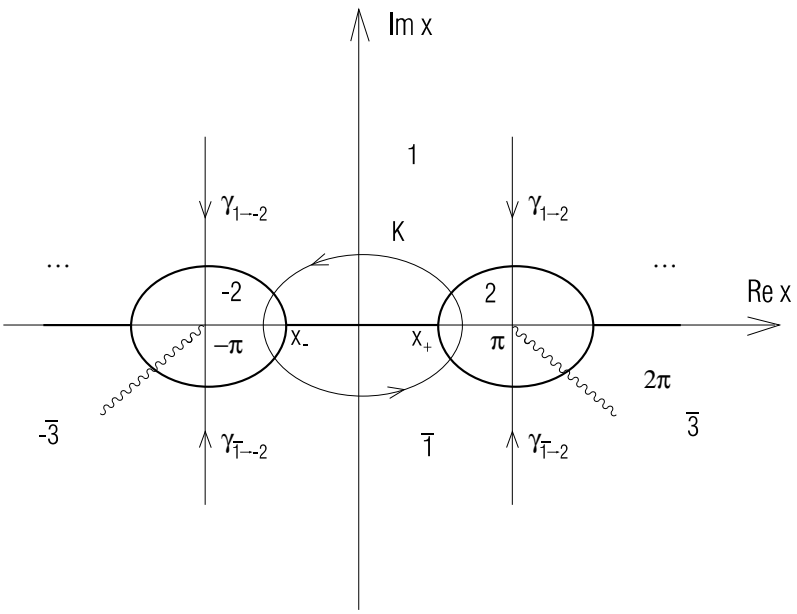

Figure 7. The Stokes graphs corresponding to formula (3.16) for another Pöshl–Teller potential.

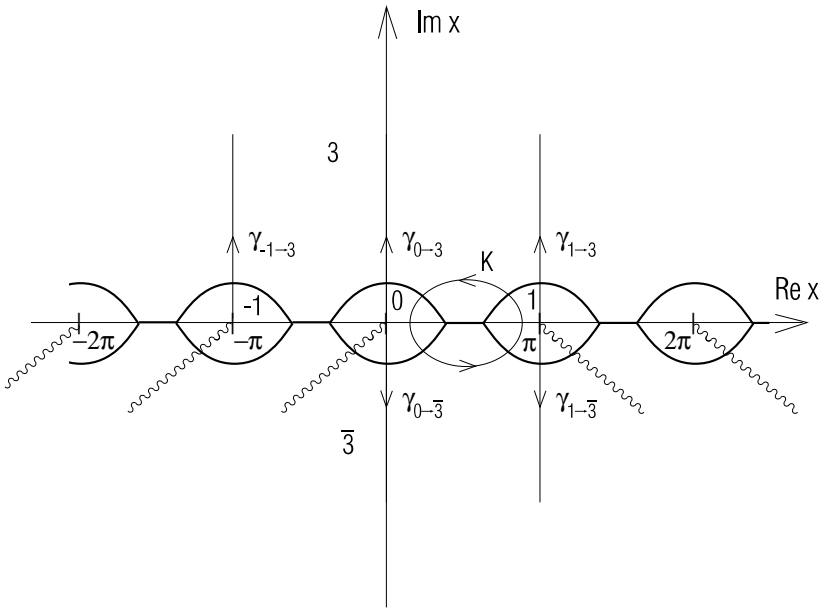

Figure 8. The Stokes graph corresponding to the exact JWKB formula (3.17).

condition for the coefficients χ in (2.7) to be related by such transformations (otherwise we would get only a relation between *different* dynamics connected by these transformations).

Suppose, therefore, that $q(x, E, \lambda)$ satisfies the following symmetry relation:

$$q(y(x), E, \lambda) = q(x, E, \lambda) \tag{2.8}$$

for $x \to y(x)$ (general properties of $y(x)$ to keep $\tilde{q}(y(x), E, \lambda)$ meromorphic have been discussed in [23]). We shall also assume that it is always possible to find (if necessary) $\delta(x, E, \lambda)$ such that (2.8) is satisfied by $\tilde{q}(x, E, \lambda)$ as well under the same transformation.

In general, $x \to y(x)$ is a variable transformation in the Schrödinger equation leading us to a new q-function as given by the following formula:

$$q(x, E, \lambda) \to q(y(x), E, \lambda)y'^2(x) + \frac{1}{\lambda^2}\left[\frac{3}{4}\frac{y''^2(x)}{y'^2(x)} - \frac{1}{2}\frac{y'''(x)}{y'(x)}\right]. \tag{2.9}$$

It is now easy to see that by using the earlier mentioned freedom in forming the $\tilde{q}$-function corresponding to a new q as given by (2.9), we can achieve the form $\tilde{q}(y(x), E, \lambda)y'^2(x)$ by removing the second term in the rhs of (2.9) and adding (if necessary) the Langer term $\frac{\delta(x)}{\lambda^2}y'^2(x)$.

If the corresponding Stokes graph is now to be invariant under such a transformation then the full set of actions (2.3) which defines this Stokes graph has to be invariant too, up to multiplicative constants. But, according to (2.8) and the above comment, we have

$$\int_{x_k}^{x} \sqrt{\tilde{q}(\xi, E, \lambda)}\,\mathrm{d}\xi = \int_{x_k}^{x} \sqrt{\tilde{q}(y(\xi), \lambda, E)}\,\mathrm{d}\xi = \int_{y(x_k)}^{y(x)} \sqrt{\tilde{q}(\xi, E, \lambda)}\frac{\mathrm{d}\xi}{y'(\xi)} \tag{2.10}$$

where, according to (2.8), $y_i = y(x_i)$ are again (other) turning points of $\tilde{q}(x, E, \lambda)$.

From (2.9) we can conclude that this action set invariance is achieved if $y'(x) = C$, where C is real.

Therefore, the allowed transformations $y(x)$ are linear. Since they constitute a group then it is easy to see that if $|C| \neq 1$ then $\tilde{q}(x, E, \lambda)$ has to have common accumulation points of

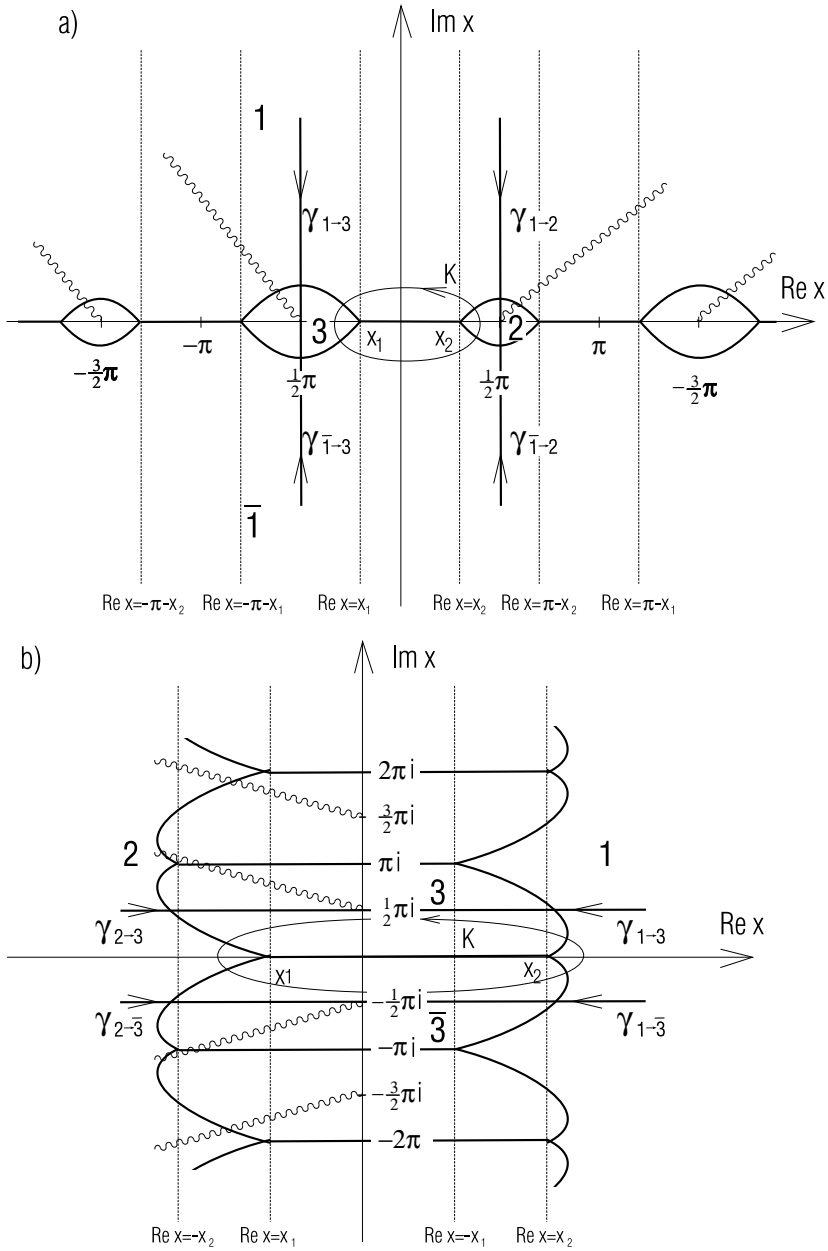

Figure 9. The Stokes graphs corresponding to the potentials $V_9(x, \lambda)$ (*a*) and $V_{10}(x, \lambda)$ (*b*), given by formulae (3.18).

their roots and poles. Therefore, we shall limit further considerations to less singular cases of $\tilde{q}(x, E, \lambda)$, which means that we shall put $C = \pm 1$. The latter limitation leaves us with only two types of allowed symmetry transformations: one which is essentially a reflection $x \to -x$ and the other a complex translation of the x-plane.

In this way we have shown that the most appropriate forms of potentials for discussing

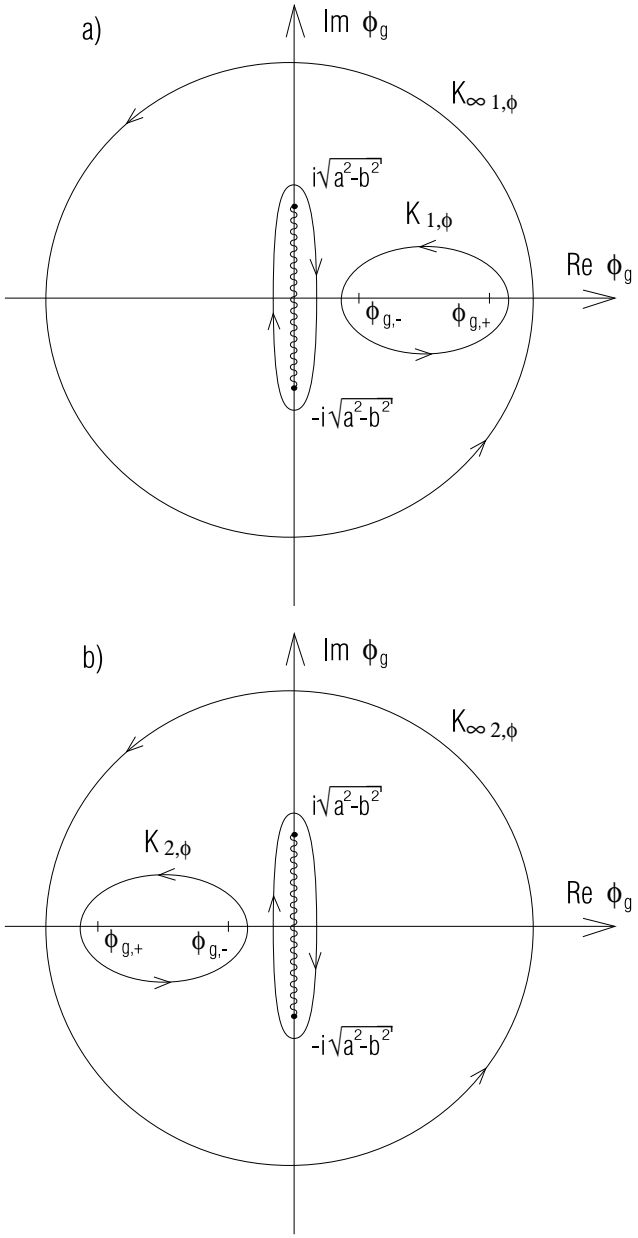

Figure 10. The two-sheeted ϕ_9-Riemann surface for the unbroken superpotential ϕ_9 (case 1^0).

their possible JWKB quantization exactness are their periodic representations. The latter can be additionally accompanied by the reflection transformation.

Therefore, in the next section, we shall consider systematically the periodic potentials only.

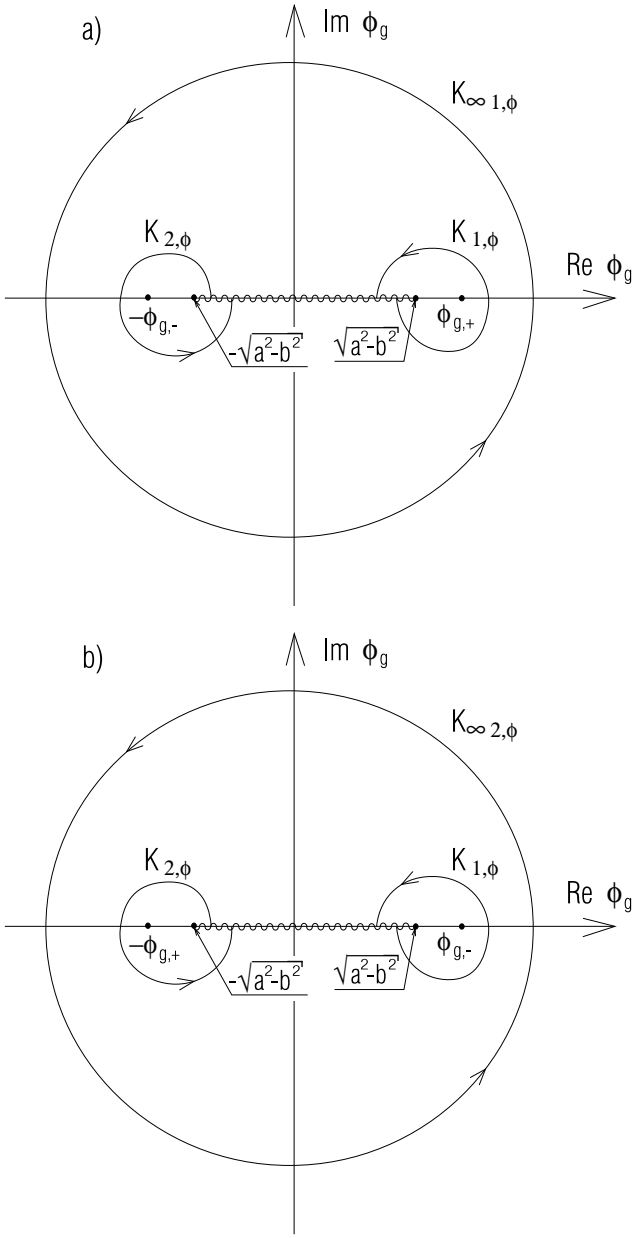

Figure 11. The two-sheeted ϕ_9-Riemann surface for broken superpotential ϕ_9 (case 2^0).

3. Periodic potentials quantized exactly by their JWKB formulae

3.1. Periodic holomorphic (entire) potentials

In general, $q(x, E, \lambda)$ as a *meromorphic* function of complex x can be periodic with at most two independent (in general complex) periods [18]. However, in the case of being *holomorphic* $q(x, E, \lambda)$ can have only one period (being a constant in the presence of the second one).

Further, since $q(x, E, \lambda)$ is assumed to be real for its real arguments then its period can be only real or only pure imaginary. For an obvious reason we shall consider only the last case, assuming for simplicity that the period is equal to $2\pi\mathrm{i}$. In this case, $q(x, E, \lambda)$ can be expanded into the following Fourier series [18] (note that for holomorphic potentials we can always put $\delta(x, E, \lambda) \equiv 0$):

$$q(x, E, \lambda) = \sum_{x=-\infty}^{\infty} q_n(E, \lambda)\mathrm{e}^{nx}. \tag{3.1}$$

If the behaviour of $q(x, E, \lambda)$ at x-infinity is to be of a finite type, series (3.1) has to be abbreviated providing us with a finite sum. The latter should contain at least three terms if we want $q(x, E, \lambda)$ to possess bound states. Let k and l $(k > l + 1)$ be, therefore, the upper and lower limits of this abbreviation, respectively. A few cases for which $k - l = 2, 3, 4$ have been considered in detail in [32]. Only one of them with $k = 2$ and $l = 0$ has been found to be quantized JWKB exactly and is considered in some detail below.

In these investigations we shall make intensive use of the Weierstrass product representation for the abbreviated series (3.1) in order to perform necessary calculation of phases of $q(x, E, \lambda)$ alone as well as its functions. This representation is considered in appendix A. There we have also explicitly calculated the relevant total phases of $q(x, E, \lambda)$ for the case $k = 2$, $l = 0$ considered just below to provide us with an example of such calculations.

Case: $k = 2, l = 0$. The case can be written as

$$q_1(x, E, \lambda) = \alpha(E, \lambda)\mathrm{e}^{2x} - 2\beta(E, \lambda)\mathrm{e}^{x} + \gamma(E, \lambda) \tag{3.2}$$

where $\alpha(E, \lambda)$, $\beta(E, \lambda)$ and $\gamma(E, \lambda)$ are known functions of E and λ. In particular, for $\alpha \equiv \beta \equiv 1$ and $\gamma \equiv -E$, we get the well known Morse potential [19].

With $\alpha, \beta, \gamma > 0$ and $\beta^2 > \alpha\gamma$ we get for $q_1(x, E, \lambda) = 0$ two real roots (modulo $2\pi\mathrm{i}$) and the corresponding Stokes graph shown in figure 2, where $x_\pm = \ln(\beta \pm \pm\sqrt{(\beta^2 - \alpha\gamma)})$. The quantization condition (2.7) according to the figure now looks as follows:

$$\exp\left[-\lambda \oint_K q_1{}^{\frac{1}{2}}(x, \lambda, E)\,\mathrm{d}x \right] = -\frac{\chi_{1\to 3}(\lambda, E)\chi_{2\to\bar{3}}(\lambda, E)}{\chi_{1\to\bar{3}}(\lambda, E)\chi_{2\to 3}(\lambda, E)}. \tag{3.3}$$

It follows from the figure that $\chi_{2\to 3} = \chi_{2\to\bar{3}} \equiv 1$ and $\chi_{1\to 3} \equiv \chi_{1\to\bar{3}}$. The first of these identities is satisfied because both the paths $\gamma_{2\to 3}$ and $\chi_{2\to\bar{3}}$ can be pushed out to infinity, whilst the second is satisfied because of the periodicity of the corresponding integrands in formulae (2.6) for $\chi_{1\to 3}$ and $\chi_{1\to\bar{3}}$. Therefore, we are left finally with the JWKB formula which gives *exact* energy levels in this case.

It should be noticed, however, that the equality of coefficients $\chi_{1\to 3}$ and $\chi_{1\to\bar{3}}$ is not immediate, i.e. it does not follow as a direct result of the periodicity of $q_1(x, E, \lambda)$. First we have to define the total phase of $q_1(x, E, \lambda)$ according to the prescriptions of appendix A in order to define uniquely its square roots present in the coefficients $\chi_{1\to 3}$ and $\chi_{1\to\bar{3}}$. This has been done in appendix A where we have found that the phases of $q_1(x, E, \lambda)$ on the integration paths of $\chi_{1\to 3}$ and $\chi_{1\to\bar{3}}$ differ exactly by 4π, i.e. by the period of the square roots of $q_1(x, E, \lambda)$ just mentioned.

It should now be stressed that to get the last result the phases of the *exponential* factor in the corresponding Weierstrass product have had to be taken into account: i.e., counting the relevant phases provided by the roots of $q_1(x, E, \lambda)$ *alone* would give us an *incorrect* result. This is what was not taken into account in the corresponding calculations of Rosenzweig and Krieger [4] and Crescimanno [11]. Another possible source of erroneous calculations of the

phases of $q_1(x, E, \lambda)$ provided by these authors could be the arbitrary way they estimated these phases which were provided by roots of $q_1(x, E, \lambda)$, spread infinitely on the complex plane. The *only* correct way of performing these calculations has to be relied on the corresponding Weierstrass product representation of $q_1(x, E, \lambda)$.

It can be shown that considering higher values of $k - l$ (>3) does not lead us to the exact JWKB formulae since in these cases the conditions used so far (two real turning points: reality and periodicity) are not sufficient to cause full cancellation of χ in the corresponding quantization conditions [32]. The main reason for that is that for higher values of $k + l$ an increasing number of complex turning points in a basic strip of periodicity causes χ entering the corresponding quantization conditions to be no longer related by the condition of periodicity, because all the relevant integrations are performed inside the same basic strip.

The above situation does not change even if we additionally make $q(x, E, \lambda)$ an even function of x.

3.2. Periodic meromorphic potentials

The reality condition demanded for $q(x, E, \lambda)$ allows us to choose its two possible basic periods: pure real and pure imaginary.

Within this class of potentials we can obviously ignore $q(x, E, \lambda)$ with a real period but without real poles. We must therefore consider the following possibilities for $q(x, E, \lambda)$:

(a) It is holomorphic in some vicinity of the real axis but meromorphic outside it and is periodic with its unique imaginary period equal to $2\pi\mathrm{i}$.
(b) It is meromorphic on the real axis with the only imaginary period equal to $2\pi\mathrm{i}$.
(c) It is meromorphic on the real axis with the only real period equal to 2π.
(d) It is meromorphic on the real axis with two periods: a real one equal to 2π and a pure imaginary one equal to $\mathrm{i}\omega$, with ω being any positive real number.

Case (a). The analysis performed in [32] showed that the following two potentials satisfying the assumption of two real roots are allowed:

$$\begin{aligned} V(x) &= \frac{\alpha_1 \mathrm{e}^x + \beta_1}{2\sinh\frac{1}{2}(x - \mathrm{i}a)\sinh\frac{1}{2}(x + \mathrm{i}a)} = \frac{\alpha_1 \mathrm{e}^x + \beta_1}{\cosh x - \cos a} \qquad a \neq \pi \\ V_4(x) &= \frac{\alpha_2 \mathrm{e}^x + \beta_2}{\cosh^2\frac{1}{2}x} \end{aligned} \tag{3.4}$$

the second of which is essentially the Rosen–Morse one [20].

To obtain from (3.4) the potentials which would have bound states some conditions on their parameters have to be satisfied. For the first potential they are

$$\alpha_1 > 0 > \beta_1 \tag{3.5}$$

with the quantized energy E varying in the following range:

$$\begin{aligned} &-\frac{2x(\alpha_1 - x)^2}{(|\alpha_1 - x) - |y|)^2 + 2|y||\alpha_1 - x|(1 - \cos a)} < E < 0 \\ &x = \sqrt{\alpha_1^2 + \beta_1^2 + 2\alpha_1\beta_1\cos a} \qquad y = \beta_1 + 2\alpha_1\cos a. \end{aligned} \tag{3.6}$$

For the second potential in (3.4), we can put $\alpha_2 > 0 > \beta_2$ without losing its generality so that the corresponding energy range is

$$-\frac{\beta_2^2}{\alpha_2 - \beta_2} < E < 0. \tag{3.7}$$

A relevant Stokes graph corresponding to the first of the potentials (3.4) is shown in figure 3. It follows immediately from the figure that the JWKB formula

$$\exp\left[-\lambda\oint_K\left[\frac{\alpha_1 e^x+\beta_1}{\cosh x-\cos a}-E\right]^{\frac{1}{2}}dx\right]=-1 \tag{3.8}$$

cannot be exact in this case since the coefficients $\chi_{1\to 4}$ and $\chi_{2\to 3}$ are not related by periodicity and do not cancel in the exact condition.

The Stokes graph for the second of the potentials (3.4) is shown in figure 4. However, in order to continue the relevant solutions corresponding to sectors 1 and 2 to sectors 3 and $\bar{3}$ (the latter two containing the second-order poles at $x=\pm\pi i$, respectively) we have to choose properly the δ-piece of ω, as defined by (2.7), to admit the integrals in (2.6) to converge at the poles. One can easily convince oneself that the choice $\delta=[4\cosh(x/2)]^{-2}$ is sufficient to achieve that aim, leaving simultaneously the original form of the Stokes graph of figure 4(*a*). unchanged: i.e., it redefines the coefficient β_2 into $\beta_2'(=\beta-1/(4\lambda)^2)$ only. Then, figure 4 provides us with the following quantization condition for the case:

$$\exp\left[-\lambda\oint_K\sqrt{\frac{\alpha_2 e^x+\beta_2-\frac{1}{16\lambda^2}}{\cosh^2\frac{1}{2}x}-E}\,dx\right]=-\frac{\chi_{1\to\bar{3}}(\lambda,E)\chi_{2\to 3}(\lambda,E)}{\chi_{1\to 3}(\lambda,E)\chi_{2\to\bar{3}}(\lambda,E)}. \tag{3.9}$$

According to appendix A the total change of the phase of $\tilde{q}_4(x,E,\lambda)$ in (3.9) is determined only by the distributions of zeros of its numerator, as well as by the corresponding zeros of $\cosh^2(x/2)$ as the denominator. Calculated (according to the rules of appendix A) with respect to the points of the lines $\operatorname{Im}x=\pi$ and $\operatorname{Im}x=-\pi$ (shifted by the period $2\pi i$) the numerator phase change amounts to 2π, which is exactly the same as the total phase change of the denominator. Therefore, the total phase change of $\tilde{q}_4(x,E\lambda)$ is exactly equal to zero: in this case, that is sufficient for $\chi_{1\to 3}$ and $\chi_{1\to\bar{3}}$, as well as for $\chi_{2\to 3}$ and $\chi_{2\to\bar{3}}$, to coincide. It means that the rhs of (3.9) is equal to -1 and the JWKB formula corresponding to (3.9) is *exact* in the case considered possessing the 'standard' Bailey form [3].

Some possible different choices of Langer corrections leading us to different, but still exact, JWKB formulae are described in [32].

Case (b). Assuming the presence of simple or second-order poles in $q(x,E,\lambda)$ it is clear that we can allow only one such pole in the main period strip. We assume its localization at $x=0$. In this way, the problem of quantization has to be reduced to a half of the real axis which we choose, without loss of generality, to be the right one. The allowed classes of potentials are [32]

$$\begin{aligned}V(x,\lambda)&=\frac{\alpha_1 e^x+\beta_1}{\sinh x}\\ V_5(x,\lambda)&=\frac{\alpha_2 e^x+\beta_2}{\sinh^2\frac{1}{2}x}.\end{aligned} \tag{3.10}$$

Reasoning, as in the previous case, we have found that the JWKB formula for the first potential cannot be exact.

For the second potential in (3.10), we can assume its parameters α_2 and β_2 (without losing generality) satisfy the following conditions:

$$\begin{aligned}&\beta_2,\alpha_2+\beta_2>0>2\alpha_2+\beta_2>\alpha_2\\ &-\frac{\beta_2^2}{\alpha_2+\beta_2}<E<4\alpha_2\end{aligned} \tag{3.11}$$

to ensure an existence of bound states in the local potential well. The shape of the potential is shown in figure 5(*a*). The potential has to be modified by the 'standard' δ-term: $\delta = (4\sinh(x/2))^{-2}$ to allow the construction of the fundamental solution at $x = 0$ which results with the change $\beta_2 \to \beta_2 + (4\lambda)^{-2}$ in the potential. The quantization condition corresponding to the Stokes graph of figure 5(*b*) now reads as

$$\exp\left[-\lambda \oint_K \left[\frac{\alpha_2 \mathrm{e}^x + \beta_2 + \frac{1}{16\lambda^2}}{\sinh^2 \frac{x}{2}} - E\right]^{\frac{1}{2}} \mathrm{d}x\right] = -\frac{\chi_{1\to\bar{2}}(E,\lambda)\chi_{3\to 2}(E,\lambda)}{\chi_{1\to 2}(E,\lambda)\chi_{3\to\bar{2}}(E,\lambda)}. \tag{3.12}$$

It follows from (3.12) and figure 5(*b*) that in this case the coefficients $\chi_{1\to 2}$ and $\chi_{1\to\bar{2}}$ cancel mutually using periodicity arguments (the phase difference produced by the numerator of $\tilde{q}_5(x, E, \lambda)$ and equal to 2π for the two integration paths $\gamma_{1\to 2}$ and $\gamma_{1\to\bar{2}}$ is cancelled by its denominator $\sinh^2(x/2)$), whilst the remaining two coefficients cancel by their reality (they are real and complex conjugated to each other). The JWKB formula which follows from (3.12) is, therefore, *exact*.

Again it is worth noting that the considered potential can be modified by the δ-function in a different way to generate at least four zeros in the basic period strip of the corresponding Stokes graph allowing the numerator of the modified $\tilde{q}_5(x, E\lambda)$ to change its phase by 4π between the earlier mentioned paths (see [32]).

The possibility of making the last modification by enlarging the number of roots in the basic period strip to four still suggests performing its completion in a different manner: namely, by adding the term coinciding exactly with the second of the potentials (3.4). Of course we also need to add the corresponding standard δ-term to the potential obtained in this way. The resulting potential does not, however, satisfy the rule of no more than two turning points in the period strip, so that the possibility of the exact JWKB-quantization condition to appear should largely depend on symmetry properties of the relevant χ-coefficients. This potential can have bound states for the following regime of its parameters (see the formula below for definition of the parameters): α, α' real and sufficiently close to zero, and $\beta, \beta' > 0$. The Stokes graph corresponding to the case is shown in figure 6 and the quantization condition related to it is

$$\exp\left[-\lambda \oint_K \left[\frac{\alpha \mathrm{e}^x + \beta + \frac{1}{16\lambda^2}}{\sinh^2 \frac{x}{2}} + \frac{\alpha' \mathrm{e}^x - \beta' - \frac{1}{16\lambda^2}}{\cosh^2 \frac{x}{2}} - E\right]^{\frac{1}{2}} \mathrm{d}x\right] = -\frac{\chi_{1\to\bar{2}}(E,\lambda)\chi_{3\to 2}(E,\lambda)}{\chi_{1\to 2}(E,\lambda)\chi_{3\to\bar{2}}(E,\lambda)}. \tag{3.13}$$

It follows from the figure that the coefficients $\chi_{3\to 2}$ and $\chi_{3\to\bar{2}}$ have to cancel mutually by periodicity, but not the remaining two: no symmetry (except the complex conjugation) relates these two coefficients. However, when $\alpha = \alpha' = 0$ the potential in (3.13) becomes invariant under the reflection $x \to -x$ and then the coefficients $\chi_{1\to 2}$ and $\chi_{1\to\bar{2}}$ are equal only by the last symmetry. (Note, however, the role played in fulfilling this symmetry by 4π of the difference between the arguments of $\tilde{q}_6(x, E, \lambda)$ corresponding to the case the latter takes on the paths $\gamma_{1\to 2}$ and $\gamma_{1\to\bar{2}}$.) Therefore, the following quantization condition:

$$\exp\left[-\lambda \oint_K \left[\frac{\beta + \frac{1}{16\lambda^2}}{\sinh^2 \frac{x}{2}} - \frac{\beta' + \frac{1}{16\lambda^2}}{\cosh^2 \frac{x}{2}} - E\right]^{\frac{1}{2}} \mathrm{d}x\right] = -1 \tag{3.14}$$

is *exact*. The potential in the above formula is of Pöschl and Teller [21].

Case (c). This case contains four potentials but only the following two are hoped to be exactly quantized JWKB [32]:

$$\begin{aligned} V_7(x,\lambda) &= \frac{\alpha\sin x+\beta}{\cos^2\frac{x}{2}} \qquad -\pi < x < \pi \\ V_8(x,\lambda) &= \frac{\alpha\sin x+\beta}{\cos^2\frac{x}{2}} + \frac{\alpha'\sin x+\beta'}{\sin^2\frac{x}{2}} \qquad 0 < x < \pi \end{aligned} \tag{3.15}$$

the second of which is essentially another of Pöschl–Teller's [21].

Consider therefore the first of them. In order to have the binding potential well we have to assume $\beta > 0$ but the choice of sign of α is arbitrary since both cases are equivalent. So we shall put $\alpha > 0$ for convenience. Next we must note, however, that asymmetry introduced into the potential by $\alpha \neq 0$ completely eliminates the possibility of using the periodicity arguments that make $\alpha = 0$ in (3.15). Once more, we have to choose δ taking it in its 'standard' form: $\delta = (4\cos(x/2))^{-2}$, obtaining the Stokes graph of figure 7. It is clear from the figure that the coefficients $\chi_{1\to-2}$ and $\chi_{1\to2}$, as well as $\chi_{\bar1\to-2}$ and $\chi_{\bar1\to2}$, are now equal using periodicity arguments. Consequently, the following JWKB-quantization formula:

$$\exp\left[-\lambda\oint_K\left[\frac{\beta+\frac{1}{16\lambda^2}}{\cos^2\frac{x}{2}}-E\right]^{\frac12}\mathrm{d}x\right]=-1 \tag{3.16}$$

is *exact*.

Again some modifications of the above formula are possible [32].

Considering the second of the potentials (3.15) we first remove asymmetry in the latter (for the same reason as discussed earlier) putting $\alpha = \alpha' = 0$ and next we notice that in the basic period strip $-\pi < \mathrm{Re}\,x \leqslant \pi$, the number of the four turning points is sufficient to make the relevant χ periodic across the strip. Therefore, the only necessary modification of the potential is the 'standard' choice for δ: i.e., $\delta = (4\sin(x/2))^{-2} + (4\cos(x/2))^{-2}$ which gives the Stokes graph shown in figure 8. We obtain the following relations from the figure: $\chi_{1\to\bar3} = \chi_{-1\to3} = \chi_{1\to3}$ and $\chi_{0\to3} = \chi_{0\to\bar3}$. The first equality in both of these equality sequences follows from the parity invariance of the potential considered, whilst the second in the first one is satisfied using periodicity arguments. Therefore, the following JWKB-quantization condition:

$$\exp\left[-\lambda\oint_K\left[\frac{\beta'+\frac{1}{16\lambda^2}}{\cos^2\frac{x}{2}}+\frac{\beta'+\frac{1}{16\lambda^2}}{\sin^2\frac{x}{2}}-E\right]^{\frac12}\mathrm{d}x\right]=-1 \tag{3.17}$$

is *exact*.

Case (d). Examples of this case are provided by elliptic functions [18]. The detailed analysis performed failed to provide us with the exact JWKB formulae [32].

3.3. Four turning points in the basic period strip

This possibility has been already considered in the particular cases of the potentials $V_6(x)$ and $V_8(x)$, shown in their exact JWKB formulae (3.16) and (3.17). The symmetric distributions of their four turning points in basic period strips guaranteed the success of these formulae to quantize exactly their energy levels.

There is still another pair of potentials with four turning points in their basic period strip quantized exactly by the JWKB formulae. They are

$$\begin{aligned} V_9(x,\lambda) &= \frac{\alpha+\beta\sin x}{\cos^2 x} \qquad -\frac{\pi}{2} < x < +\frac{\pi}{2} \qquad \alpha > \beta > 0 \\ V_9^{\min} &= \tfrac{1}{2}\left(\sqrt{\alpha^2-\beta^2}+\alpha\right) \\ V_{10}(x,\lambda) &= \frac{\alpha+\beta\sinh x}{\cosh^2 x} \qquad -\infty < x < +\infty \qquad \beta > 0 \\ V_{10}^{\min} &= -\tfrac{1}{2}\left(\sqrt{\alpha^2+\beta^2}-\alpha\right). \end{aligned} \tag{3.18}$$

One can easily convince oneself that completed by the 'standard' δ-terms $((2\cos x)^{-2}$ for the first potential and $-(2\cosh x)^{-2}$ for the second one), the energy levels of both the potentials are also *exactly* JWKB quantized. For the first potential, this fact follows as a result that this potential is symmetric with respect to the real axis on the vertical lines $\operatorname{Re} x = \pm\pi/2$, whilst for the second potential it is due to the analogous symmetry which is valid on the lines $\operatorname{Im} x = \pm\pi/2$. These symmetries cause the coefficients χ calculated on the paths shown in figure 9 to cancel mutually in the corresponding quantization formulae (2.7).

4. Aperiodic exactly JWKB-quantized potentials

The periodic potential (3.2) which provides us with the exact JWKB-quantization formulae (3.3) can also serve as the source of aperiodic exactly JWKB-quantized potentials. The latter can be obtained from the former by a trivial change-of-variable procedure $x \to y(x)$ resulting in the potential transformations [23, 27] given by formula (2.9). The only necessary requirement for a relevant change is to provide by it in the resulting potentials a free (i.e. x-independent) term which can play the role of an energy parameter. The latter demand, when applied to potential (3.2), permits the following two possibilities: $1^0, \mathrm{e}^x \to x$ and $2^0, \mathrm{e}^{x/2} \to x$. Adjusting properly α, β and γ in (3.2) we get in this way the radial parts of the Coulomb potential $V_2(x)$ in the first case and of the 3D homogeneous harmonic oscillator potential $V_3(x)$ in the second one.

The same method can be applied to the potentials which are *not* exactly JWKB quantized providing us with aperiodic potentials with the same property: i.e., the method does not allow us to make any further estimations of the resulting potentials for their not being quantized exactly by the corresponding JWKB formulae (which on their own would not be a simple task).

As mentioned earlier, there is only one known aperiodic potential for which the simplest *vanishing* mechanism of all the integrations in (2.5) works. This is the harmonic oscillator potential: $V_{11}(x) = \alpha^2 x^2$, unique among all the polynomial potentials for which the corresponding JWKB formulae can be exact. This well known exactness can be easily proved using the polynomial form of the potential [32]. The same result can be achieved using its periodic representation by changing $x \to \sinh\frac{x}{2}$. However, for this particular case, the periodic representation does not show its advantage over the aperiodic one. Therefore, we shall not consider the corresponding equivalent periodic formulation.

5. More general exactly JWKB-quantized potentials

The periodic potentials considered in the previous section are the simplest ones of all the potentials quantized exactly by the JWKB formula. A generalization of their forms to the ones

which still can keep the exactness of the corresponding JWKB formula can be done in the following way.

Let $V(x)$ mean any exactly JWKB-quantized periodic potential of the previous section. Let $V(x, p)$ mean a real parameter family of periodic (with respect to x) potentials with the property that in the limit $p \to 0$, $V(x, p)$ smoothly approaches $V(x)$. Then, for p small enough, the Stokes graph corresponding to $V(x, p)$ has to resemble the Stokes graph corresponding to $V(x)$. By such a resemblance we mean the following:

(1) To any singular point of a Stokes graph of $V(x)$ there corresponds a set of singular points of a Stokes graph of $V(x, p)$ which reduce to the former point in the limit $p \to 0$. Each member of such a set we shall call a singular point blob.
(2) To any turning point of the Stokes graph of $V(x)$ there corresponds a set of turning points of $V(x, p)$ which reduces to the former point in the limit $p \to 0$. We shall call such a set a turning point blob.
(3) There is one-to-one correspondence between the sectors of the two Stokes graphs such that the sectors of a Stokes graph of $V(x, p)$ reduce smoothly to the corresponding sectors of a Stokes graph of $V(x)$ when $p \to 0$. In particular, the boundary conditions are formulated for both the potentials $V(x, p)$ and $V(x)$ in sectors satisfying the correspondence just described.
(4) Each set of Stokes lines which emerge from some singular (turning) point blob can be mapped into a definite set of Stokes lines of a Stokes graph corresponding to $V(x)$, emerging from the point to which this singular (turning) point blob reduces in the limit $p \to 0$. Each of such sets can be divided into disjoint subsets (sheaves) of Stokes lines each transforming smoothly when $p \to 0$ into one particular Stokes line emerging from the limiting point.
(5) For any p a set of symmetries of $V(x, p)$ and of its Stokes graph is the same as for $V(x)$ and its Stokes graph.

Examples of $V(x, p)$ with properties 1^0–5^0 and the Planck constant $\hbar$ as parameter p can be found in [1]. With these properties, $V(x, p)$ provides us with the JWKB formula quantizing exactly the energy levels of $V(x, p)$.

6. JWKB and SJWKB formula exactness and its relation with shape invariance symmetry of potentials

In connection with the SUSY formulation of quantum mechanics, the SUSY JWKB approximations have been suggested as different from the conventional ones which have appeared to be exact [7, 8]. It has been also noticed, however, that their exactness has been parallel to the exactness of the conventional ones [7, 8, 11].

So far, we have shown that the exactness of the conventional (i.e. not SUSY) JWKB formulae was rather exceptional and was related to the simplest singularity and turning point structures of the corresponding Stokes graphs. Since the SUSY quantum mechanics quantization problems seem to be governed by the same rules, we can expect that the exactness of the SUSY JWKB formulae have to follow in some way from the conventional ones. We will show below that this is indeed the case.

Note, however, that there is also a common conviction that the SUSY JWKB exact quantization conditions are not only independent of the conventional ones but also that their exactness in some cases of potentials is in contrast with the approximate character in these cases of the conventional JWKB formulae. As such, potentials are considered as shape invariant [29].

It is shown below that also in these cases the parallelness of the exactness of both kinds of formulae is still maintained.

Let us examine first the question of how the SUSY JWKB exact formulae follow from the conventional ones.

6.1. Exactness of SUSY JWKB formulae following from conventional ones

Let us remind ourselves that if a potential $V(x,\lambda)$ can be put in its SUSY form $V(x,\lambda) \equiv V_-(x,\lambda) = \phi^2(x,\lambda) - \phi'(x,\lambda)/\lambda + \epsilon_0$ (ϵ_0 is the energy of the fundamental level in $V(x)$ if SUSY is exact) then the conventional JWKB-quantization condition:

$$-\lambda \oint_K \sqrt{V(x,\lambda) + \frac{\delta(x,\lambda)}{\lambda^2} - E}\,\mathrm{d}x = (2m+1)\pi\mathrm{i} \qquad m = 0, 1, 2, \ldots \tag{6.1}$$

for the exact SUSY is to be substituted by [7, 8]

$$-\lambda \oint_K \sqrt{\phi^2(x,\lambda) - (E-\epsilon_0)}\,\mathrm{d}x = 2\pi\mathrm{i}m \qquad m = 0, 1, 2, \ldots. \tag{6.2}$$

If (6.1) is exact, then as previously mentioned, (6.2) is also very frequently. Let us analyse how this can happen. The analysis shall be performed for both cases of broken and unbroken superpotentials ϕ which can represent $V(x,\lambda)$. It will be shown that in both cases condition (6.2), if it is *exact*, remains the same, which is in contrast with its form representing the lowest JWKB approximation only, in which case its rhs coincides rather with (6.1), whilst the unbroken one coincides with (6.2) [7, 8, 26].

Let us now recapitulate all 11 potentials $V_k(x)$ and the corresponding $\tilde{q}_k(x, E, \lambda)$-functions we found in the previous section to be quantized exactly by the corresponding JWKB formulae. They are

$$\tilde{q}_1(x,E,\lambda) = V_1(x) - E = \alpha^2\mathrm{e}^{2x} - 2\beta\mathrm{e}^x - E$$
$$-\infty < x < +\infty \qquad \beta > 0 > E$$

$$\tilde{q}_2(x,E,\lambda) = V_2(x) + \frac{1}{4\lambda^2x^2} - E = -\frac{\alpha}{x} + \frac{\beta + \frac{1}{4\lambda^2}}{x^2} - Ex, \alpha, \beta > 0 > E$$

$$\tilde{q}_3(x,E,\lambda) = V_3(x) + \frac{1}{4\lambda^2x^2} - E = \alpha^2x^2 + \frac{\beta + \frac{1}{4\lambda^2}}{x^2} - E \qquad x, \beta, E > 0$$

$$\tilde{q}_4(x,E,\lambda) = V_4(x) - \frac{1}{(4\lambda\cosh\frac{x}{2})^2} - E = \frac{\alpha\mathrm{e}^x - \beta - \frac{1}{16\lambda^2}}{\cosh^2\frac{x}{2}} - E$$
$$-\infty < x < +\infty \qquad \beta > 0 \qquad -\beta < 2\alpha$$

$$\tilde{q}_5(x,E,\lambda) = V_5(x) + \frac{1}{(4\lambda\sinh\frac{x}{2})^2} - E = \frac{\alpha\mathrm{e}^x + \beta + \frac{1}{16\lambda^2}}{\sinh^2\frac{x}{2}} - E$$
$$0 < x < +\infty \qquad \beta, \alpha+\beta > 0 > 2\alpha+\beta > \alpha$$

$$\tilde{q}_6(x,E,\lambda) = V_6(x) + \frac{1}{(4\lambda\sinh\frac{x}{2})^2} - \frac{1}{(4\lambda\cosh\frac{x}{2})^2} - E = \frac{\beta + \frac{1}{16\lambda^2}}{\sinh^2\frac{x}{2}} - \frac{\alpha + \frac{1}{16\lambda^2}}{\cosh^2\frac{x}{2}} - E$$
$$0 < x < +\infty \qquad \alpha, \beta > 0$$

$$\tilde{q}_7(x,E,\lambda) = V_7(x) + \frac{1}{(4\lambda\cos\frac{x}{2})^2} - E = \frac{\alpha + \frac{1}{16\lambda^2}}{\cos^2\frac{x}{2}} - E \tag{6.3}$$
$$-\pi < x < \pi \qquad \alpha > 0$$

$$\tilde{q}_8(x,E,\lambda) = V_8(x) + \frac{1}{(4\lambda\cos\frac{x}{2})^2} + \frac{1}{(4\lambda\sin\frac{x}{2})^2} - E = \frac{\alpha + \frac{1}{16\lambda^2}}{\cos^2\frac{x}{2}} + \frac{\beta + \frac{1}{16\lambda^2}}{\sin^2\frac{x}{2}} - E$$

$$\tilde{q}_9(x, E, \lambda) = V_9(x, \lambda) - \frac{1}{(2\lambda \cosh x)^2} - E$$
$$= \frac{\alpha - \frac{1}{4\lambda^2} + \beta \sinh x}{\cosh^2 x} - E \qquad -\infty < x < +\infty \qquad \beta > 0$$
$$0 < x < \pi \qquad \alpha, \beta > 0$$
$$V_9^{\min} = -\tfrac{1}{2}\left(\sqrt{\alpha^2 + \beta^2} - \alpha\right)$$
$$\tilde{q}_{10}(x, E, \lambda) = V_{10}(x, \lambda) + \frac{1}{2\lambda^2} - E$$
$$= \frac{\alpha + \frac{1}{4\lambda^2} + \beta \sin x}{\cos^2 x} - E \qquad -\frac{\pi}{2} < x < +\frac{\pi}{2} \qquad \alpha > \beta > 0$$
$$V_{10}^{\min} = \tfrac{1}{2}\left(\sqrt{\alpha^2 - \beta^2} + \alpha\right) \qquad \tilde{q}_{11}(x, E, \lambda) = V_{11}(x, \lambda) - E = \alpha^2 x^2 - E.$$

In order to represent the above potentials by their SUSY ones, one has, in principle, to solve the non-uniform Riccati equations with their rhs given by the potentials listed. In general such a task is rather difficult. Fortunately, for most of the above potentials, it is possible to find these representations just by a trivial guess. To each of the potentials listed above one can guess several (at least two) solutions, one of which corresponds to a superpotential realizing the SUSY exactly whilst the remaining ones correspond to a broken supersymmetry. The latter means that the supersymmetry breaking can be realized in many ways. The ways considered below take into account only the possibility to define by a superpotential ϕ the corresponding ground state solution Ψ_0 by the following representation:

$$\Psi_0(x) = \exp\left[-\lambda \int^x \phi(y)\,\mathrm{d}y\right] \qquad a < x < b \tag{6.4}$$

where a, b $(a < b)$ define boundaries of the corresponding quantization problem. Note that Ψ_0, as given by (6.4), satisfies the Schrödinger equation (2.1) for $E = \epsilon_0$ with the potentials $V(x, \lambda)(\equiv V_-(x, \lambda))$ listed above. There are four possibilities:

(1^0) Ψ_0 vanishes at both the boundaries a, b—the supersymmetry is exact and Ψ_0 is the ground state wavefunction;

(2^0) and (3^0) Ψ_0 vanishes at one of the boundaries only (a or b respectively)—the supersymmetry *has* to be broken; and

(4^0) Ψ_0 blows up at both the boundaries—the supersymmetry seems essentially to be broken but there is still a possibility that the ground state Ψ_0 has been constructed by the erroneous choice of ϕ—there are infinitely many solutions satisfying the Schrödinger equation considered with $E = \epsilon_0$ but blowing up at both the boundaries even if the corresponding ground state exists with this energy.

The latter possibility cannot happen in cases 2^0 and 3^0: blowing up of Ψ_0 at one of the boundaries only means that the ground state with $E = \epsilon_0$ cannot exist in these cases. One can expect, therefore, that resulting relations between the energy spectra provided by the quantization conditions defined by the allowed superpotentials ϕ_k, corresponding to each of the potentials V_k, $k = 1, \ldots, 11$, listed earlier, and the original spectra of the latter potentials can depend on the way the supersymmetry is broken by each particular ϕ_k.

A full list of the allowed superpotentials ϕ_k corresponding to each of the potentials V_k, $k = 1, \ldots, 11$, with the above properties 1^0–4^0 (attaching to each of them the corresponding category) are collected in appendix B.

Let us note further that because λ can vary we can take it sufficiently large to expand the integrand in (6.2) into a series with respect to $\phi' - \delta/\lambda$. We get

$$-\lambda \oint_K \sqrt{\phi^2 - \tilde{E}}\,\mathrm{d}x - \sum_{n\geqslant 1} \frac{1}{\lambda^{n-1}} \frac{\Gamma[n-\frac{1}{2}]}{n!\Gamma[-\frac{1}{2}]} \times \oint_K \frac{[\phi' - \frac{\delta}{\lambda}]^n}{(\phi^2 - \tilde{E})^{n-\frac{1}{2}}}\,\mathrm{d}x = (2m+1)\pi\mathrm{i} \tag{6.5}$$

where $\tilde{E} = E - \epsilon_0$.

Making a further change of variable: $x \to \phi = \phi(x, \lambda)$ in the integrands of the series in (6.5) and putting $F_1(\phi, \lambda) \equiv \phi'(x(\phi, \lambda), \lambda) - \delta(x(\phi, \lambda), \lambda)/\lambda$ and $F_2(\phi, \lambda) \equiv \phi'(x(\phi, \lambda), \lambda)$ we obtain

$$-\lambda \oint_K \sqrt{\phi^2 - \tilde{E}}\,\mathrm{d}x - \sum_{n\geqslant} \frac{1}{\lambda^{n-1}} \frac{\Gamma[n-\frac{1}{2}]}{n!\Gamma[-\frac{1}{2}]} \times \oint_{K_\phi} \frac{F_1^n(\phi, \lambda)}{(\phi^2 - \tilde{E})^{n-\frac{1}{2}}} \frac{\mathrm{d}\phi}{F_2(\phi, \lambda)} = (2m+1)\pi \tag{6.6}$$

where the integrations under the sum in (6.6) go now into the ϕ-plane. A possibility of making these integrations now depends on analytic properties of the functions $F_{1,2}(\phi, \lambda)$ on the complex ϕ-plane. The explicit forms of these functions for each of the potentials of (6.3) are also given in appendix B. The following basic observations of the properties of $F_{1,2}(\phi, \lambda)$ are valid for the integrations in (6.6):

(1) In all the cases considered the functions $F_{1,2}(\phi, \lambda)$ are *holomorphic* being defined on at most two-sheeted ϕ-Riemann surfaces $\boldsymbol{R}_\phi$ with square root branch points.
(2) There are as many different sheets of $\boldsymbol{R}_\phi$ as the number of different singularities of the considered potentials mapped by $\phi(x, \lambda, a)$ into different infinities of the corresponding ϕ-Riemann surfaces.
(3) On each of these sheets the functions $F_{1,2}(\phi, \lambda)$ diverge to infinity no faster than ϕ^2 when $\phi \to \infty$.

The above properties of $F_{1,2}(\phi, \lambda)$ can be easily understood noticing that $\phi(x, \lambda, a)$ and its derivative $\phi'(x, \lambda, a)$, as a function of x, are meromorphic at the same points as the potentials defining them. (Note, however, that as such we also consider points located at infinities, so that, for example, e^x or x^n are considered to be singular for $x \to \infty$.) Therefore, these singularities are all mapped by the transformation $x \to \phi = \phi(x, \lambda)$ into the corresponding infinities of the ϕ-Riemann surface sheets. If these mappings differ in some way then the corresponding infinities have to be approached on different sheets of $\boldsymbol{R}_\phi$.

On the other hand, the singular points of $F_{1,2}(\phi, \lambda)$ can appear only at the points where $\phi'(x, \lambda, a)$ vanishes. These are exactly the square root branch points of $F_{1,2}(\phi, \lambda)$ on their corresponding ϕ-Riemann surfaces since generic zeros of $\phi'(x, \lambda, a)$ are simple.

The second of the above properties follows as a result of the specific singularities the considered potentials have on the x-plane. Namely, one can easily check that by taking into account all types of singularities of these potentials we have

(i) For $V_-(x, \lambda)$ diverging as e^x for $x \to \infty$, $F_{1,2}$ diverge linearly with ϕ when $\phi \to \infty$ on a given sheet.
(ii) For $V_-(x, \lambda)$ diverging as x^2 for $x \to \infty$, $F_{1,2}$ approach constant values when $\phi \to \infty$.
(iii) For x close to a second-order pole x_0 of $V_-(x, \lambda)$, $F_{1,2}$ diverge to infinity as ϕ^2 when $\phi \to \infty$ on a sheet, the infinity of which is a map of the corresponding pole x_0.

Next we can still observe the following crucial properties of the transformation $x \to \phi = \phi(x, \lambda)$ and of the quantization conditions (6.1) for all the exactly JWKB-quantized potentials considered above:

(4) In the cases when there are two pairs of turning points in the corresponding basic period strip which can be used equivalently to quantize the energy levels by (6.1) then $x \to \phi = \phi(x, \lambda)$ maps these pairs and the corresponding pieces of the basic period strip into corresponding sheets of the ϕ-Riemann surfaces in such a way that conditions (6.1) can be written independently and equivalently on each sheet.

Now taking into account the above properties (1)–(4) we can rewrite the contour integrals of the series in (6.6) as the following sum over the contours distributed on the different sheets:

$$-\lambda \oint_K \sqrt{\phi^2 - \tilde{E}}\, \mathrm{d}x - \sum_{n \geqslant 1} \frac{1}{\lambda^{n-1}} \frac{\Gamma[n - \frac{1}{2}]}{n!\Gamma[-\frac{1}{2}]} \times \frac{1}{2} \sum_{r=1,2} \oint_{K_{r,\phi}} \frac{F_1^n(\phi, \lambda)}{(\phi^2 - \tilde{E})^{n-\frac{1}{2}}} \frac{\mathrm{d}\phi}{F_2(\phi, \lambda)} = (2m + 1)\pi \tag{6.7}$$

where $K_{r,\phi}$, $r = 1, 2$ surround the corresponding pairs of turning points.

Since $F_{1,2}(\phi, \lambda)$ are holomorphic outside the contours $K_{r,\phi}$, $r = 1, 2$ we can deform these contours to the ones which partly surround the cuts on each sheet and partly coincide with the circles of sufficiently large radii completing these contours on each sheet. Then the integrations along the cuts cancel pairwise and we are left only with the integrations along the circles each taken on different sheets.

Next expanding the denominators on these circles we get

$$-\lambda \oint_K \sqrt{\phi^2 - \tilde{E}}\, \mathrm{d}x - \sum_{n \geqslant 1} \frac{1}{\lambda^{n-1}} \frac{\Gamma[n - \frac{1}{2}]}{n!\Gamma[-\frac{1}{2}]} \times \frac{1}{2} \sum_{r=1,2} \sum_{k \geqslant 0} \tilde{E}^k \frac{\Gamma[k + n - \frac{1}{2}]}{k!\Gamma[n - \frac{1}{2}]} \oint_{C_{r,\phi}} \frac{F_1^n(\phi, \lambda)}{\phi^{2n+2k-1}} \frac{\mathrm{d}\phi}{F_2(\phi, \lambda)} = (2m + 1)\pi \mathrm{i}. \tag{6.8}$$

A final result of the integrations in (6.8) depends now of course on the particular forms of the expansions of $F_{1,2}(\phi, \lambda)$ into their corresponding Laurent series.

It can be easily checked, however, that the series in the lhs of (6.8) becomes energy independent only in the case when the Laurent series expansions of $F_{1,2}$ both abbreviate at a power of ϕ no higher than second. This is simply the case of the potentials considered.

Suppose, therefore, that on the rth sheet $F_{1,2}(\phi, \lambda) = \sum_{k \geqslant 1} F^{(r)}_{1,2;k}(\lambda)\phi^{-k} + a^{(r)}_{1,2}(\lambda) + b^{(r)}_{1,2}(\lambda)\phi + c^{(r)}_{1,2}(\lambda)\phi^2$. Then from (6.8) we get

$$-\lambda \oint_K \sqrt{\phi^2 - \tilde{E}}\, \mathrm{d}x + \frac{1}{2} \sum_{r=1,2} \left(\pi \mathrm{i} \delta_{b^{(r)}0} \delta_{c^{(r)}0} + \pi \mathrm{i} \delta_{c^{(r)}0} - 2\pi \mathrm{i} \frac{\lambda}{c_2^{(r)}} \left[\sqrt{1 - \frac{c_1^{(r)}}{\lambda}} - 1 \right] \right) = (2m + 1)\pi \mathrm{i}. \tag{6.9}$$

As can be seen from (6.9), the contributions of the constant and the linear divergences of $F_{1,2}(\phi, \lambda)$ at the ϕ-infinities are completely independent of particularities of these divergences. One can easily check, however, that this is also true for the contributions of the corresponding coefficients $c^{(r)}_{1,2}(\lambda)$: i.e., these contributions are also independent of both the potential considered and the coefficients themselves, depending instead entirely on the type of the potential singularities (which for the case considered correspond to the second-order poles of the potentials): namely, these contributions are *always* the same giving the value $\pi \mathrm{i}$ for each circle integration independent of whether the supersymmetry is exact or broken [32].

Therefore, for the 11 potentials (6.3) we have proved in this way the following equality between the JWKB integral and its SUSY form:

$$\oint_K \sqrt{\phi^2(x,\lambda) - \frac{1}{\lambda}\phi'(x,\lambda) + \frac{\delta(x,\lambda)}{\lambda^2} - (E-\epsilon_0)}\,\mathrm{d}x = \oint_K \sqrt{\phi^2(x,\lambda) - (E-\epsilon_0)}\,\mathrm{d}x - \frac{\pi\mathrm{i}}{\lambda}. \tag{6.10}$$

Of course, we could perform the above calculations in the reverse direction: i.e. from SUSY to the conventional JWKB integral, which can formally be realized in formula (6.10) by moving the term $\pi\mathrm{i}$ from the rhs to the lhs of the formula.

In a condensed form, the method of performing the above calculations has been demonstrated for the potential $V_9(x)$ in appendix C.

The final conclusion is very important and, furthermore, allows for the following generalization of (6.10):

$$\oint_K \sqrt{\phi^2(x,\lambda) \pm \frac{1}{\lambda}\phi'(x,\lambda) + \frac{\delta(x,\lambda)}{\lambda^2} - (E-\epsilon_0)}\,\mathrm{d}x = \oint_K \sqrt{\phi^2(x,\lambda) - (E-\epsilon_0)}\,\mathrm{d}x \pm \frac{\pi\mathrm{i}}{\lambda}. \tag{6.11}$$

Also, the entire discussion above allows us to formulate a slightly more general theorem which includes as its particular cases the 11 potentials listed by (6.3). Namely, we have the following theorem.

Theorem 1. *Let the following assumptions be satisfied for the potential $V(x,\lambda,a)$ and its superpotential partner $\phi(x,\lambda,a)$:*

(a) $V(x,\lambda,a)$ and $\phi(x,\lambda,a)$ are meromorphic on the x-plane.
(b) The functions $F_{1,2}(\phi,\lambda,a)$ corresponding to the superpotential ϕ are holomorphic *on a n-sheeted ϕ-Riemann surface $\boldsymbol{R}_\phi$ with the square root branch points.*
(c) At the infinity of each sheet the functions $F_{1,2}(\phi,\lambda,a)$ diverge with an integer power of ϕ but no faster than the second.
(d) There are pairs of turning points (among which there is a pair of physical ones) such that the JWKB (SJWKB) quantization integral can be written alternatively and equivalently around any of the pairs and on each sheet of $\boldsymbol{R}_\phi$ there is an image of exactly one *of these pairs of turning points.*

Then relation (6.11) holds.

The proof of this theorem differs from the earlier proof of formula (6.11) only by the assumed number of sheets of the Riemann surface $\boldsymbol{R}_\phi$ which is finite but not limited to two.

Let us finish this section by comparing the energy levels obtained by formula (6.1) with those obtained by (6.2) using in the latter the respective superpotentials of cases 1^0–4^0 above. We can conclude that the levels given by (6.1) are reproduced by (6.2):

(i) Exactly in case 1^0 of the superpotentials.
(ii) By being *shifted up* by *half* a unit used to enumerate the levels in cases 2^0 and 3^0 of the superpotentials.
(iii) By being *shifted up* by a *whole* unit used to enumerate the levels in cases 2^0 and 3^0 of the superpotentials.

It is clear that the above differences follow as a result of the different enumeration of energy levels in the compared spectra (m in (6.1) starts from zero whilst in (6.2) from unity), as well as due to different choices of the energy levels ϵ_0 with respect to which the levels of the spectra are measured in all of the cases 1^0–4^0.

6.2. Exactness of SUSY and conventional JWKB quantizations in the case of shape invariant potentials

In the previous section we have shown that for the class of potentials given by (6.3) for which the JWKB formulae were exact the corresponding SUSY partner formulae were exact also.

The exactness of the JWKB formulae obtained by detailed analysis of the potentials and the corresponding Stokes graphs was dependent on particular symmetries of the latter and on the singularity structure of the potentials.

The exactness of the SJWKB formulae followed as a direct result of analytic (or rather meromorphic) properties of the potentials mapped into the Riemann surface of the superpotentials themselves, i.e. *no other* additional properties of the potentials and superpotentials have been necessary for this simultaneous exactness of formulae (6.1), (6.2).

However, the exactness of the SJWKB quantization formulae for the same potentials (6.3) has been argued to also follow as a result of their common property of being shape invariant [28–31]. This means that $V_k(x,\lambda) \equiv V_{k,-}(x,\lambda), k = 1,\dots,11$, depends additionally on some parameter a so that for its SUSY partner $V_{k,+}(x,\lambda,a)$ we have [29]

$$\begin{aligned} &V_{k,+}(x,\lambda,a_1) = V_{k,-}(x,\lambda,a) + R_k(a_1) \\ &k = 1,\dots,11 \end{aligned} \tag{6.12}$$

with $a_1 = f_k(a)$.

It appears that all the potentials considered so far and being exactly JWKB (SJWKB) quantized belong to the class of the shape invariant potentials [28]. In their case each $f(a)$ is simply a translation of the parameter a.

The exactness of (6.2) following from (6.12) has been suggested by Dutt *et al* [30] and established by Barclay and Maxwell [31] at the perturbative level. It was argued also (see Cooper *et al* and [28], for example) that the exactness of SJWKB formulae (6.2) which follows from (6.12) takes place even when the conventional one fails.

As we have seen earlier, the latter claim, however, cannot be true in the case of the 11 potentials (6.3) and in the case of potentials satisfying the conditions of theorem 1.

Nevertheless, it is still interesting how the shape invariance symmetry expressed by (6.12) is related to the singularity structures of the potentials (6.3) which guaranteed the exactness of their corresponding JWKB and SJWKB formulae.

The following theorem establishes this relation.

Theorem 2. *Let the following assumptions be satisfied for the potential $V(x,\lambda,a)$:*

(1) The potential $V(x,\lambda,a)$ is shape invariant.
(2) The potential $V(x,\lambda,a)$ satisfies all the conditions of theorem 1.

Then the conventional and SJWKB formulae for $V_\pm(x,\lambda)$ are exact.

Proof. The theorem follows from the repetitions of the reasonings which lead us to theorem 1 and from the following sequence of equalities:

$$\begin{aligned} \oint_K (\phi^2(a) - \tilde{E})^{\frac{1}{2}}\,\mathrm{d}x &= \oint_K (\phi^2(a_1) - \tilde{E} + R(a_1))^{\frac{1}{2}}\,\mathrm{d}x \\ &\quad + \oint_K (f(F_1^-(a_1), \tilde{E} - R(a_1)) - f(F_1^+(a), \tilde{E}))\,\mathrm{d}x = \cdots \\ &= \oint_K (\phi^2(a_m) - \tilde{E} + R(a_1) + \cdots + R(a_m))^{\frac{1}{2}}\,\mathrm{d}x \end{aligned}$$

$$+\sum_{p=1}^{m}\oint_K (f(F_1^-(a_p),\tilde{E}-R(a_1)-\cdots-R(a_p))$$
$$-f(F_1^+(a_{p-1}),\tilde{E}-R(a_1)-\cdots R(a_{p-1})))\,\mathrm{d}x \quad a_0=a \quad R(a_0)=0 \tag{6.13}$$

where $f(F_1^{\pm},\tilde{E})$ is defined by

$$\oint_K\left[\phi^2(a)\pm\frac{1}{\lambda}\phi'(a)+\frac{\delta}{\lambda^2}-\tilde{E}\right]^{\frac{1}{2}}\mathrm{d}x=\oint_K(\phi^2(a)-\tilde{E})^{\frac{1}{2}}\mathrm{d}x+\oint_K f(F_1^{\pm}(a),\tilde{E})\,\mathrm{d}x \tag{6.14}$$

with $F_1^{\pm}=\pm\phi'+\delta/\lambda$.

From assumption (2) it follows that every one of the contour integrals in the sum of the rhs of (6.13) being rewritten to be taken on some sheet of R_ϕ can be taken on *each* sheet of R_ϕ in the following way:

$$\oint_K f(F^{\pm}(x,a),\tilde{E})\,\mathrm{d}x=\frac{1}{n}\sum_{r=1}^{n}\oint_{K_{\phi,r}} f(F_1^{\pm}(\phi,a),\tilde{E})\frac{\mathrm{d}\phi}{F_2(\phi)} \tag{6.15}$$

where $F_2(\phi)\equiv\phi'(x(\phi))$.

It follows further from assumption (2) that every contour $K_{\phi,r}$, $r=1,\ldots,n$, can be deformed on a sheet which is defined onto a circle of sufficiently large radius and to pieces of this contour which cancel mutually with analogous pieces of other contours. The net result of these deformations are the integrations performed on every sheet along the circles with sufficiently large radii. On the circles the integrated f (divided by F_2) are holomorphic and diverging to infinity no faster than the second power of ϕ. This guarantees that all these integrals can be calculated in a way similar to that which we used earlier in proving theorem 1. In particular, independent of the type of singularity of $V(x,\lambda,a)$ each infinity contributes the same to the sum (6.15): namely $\pm\mathrm{i}\pi/\lambda$ for the $F_1^{\pm}$ cases, respectively. Therefore, the total value of the integral in the lhs of (6.15) is also $\pm\mathrm{i}\pi/\lambda$, accordingly. Finally, formula (6.13) becomes

$$\oint_K(\phi^2(a)-\tilde{E})^{\frac{1}{2}}\,\mathrm{d}x=\oint_K(\phi^2(a_m)-\tilde{E}+R(a_1)+\cdots+R(a_m))^{\frac{1}{2}}\,\mathrm{d}x-2\pi\mathrm{i}m \qquad a_0=a \qquad R(a_0)=0. \tag{6.16}$$

Substituting now in (6.16) $\tilde{E}=R(a_1)+\cdots+R(a_m)\equiv\tilde{E}_m$ we get the result (6.2) where for the broken supersymmetry the integer m begins instead from $m=1$.

The corresponding conventional JWKB exactness follows immediately from equalities (6.11) and (6.14). Namely, we get

$$\oint_K\left[\phi^2(a)\pm\frac{1}{\lambda}\phi'(a)+\frac{\delta}{\lambda^2}-R(a_1)-\cdots-R(a_m)\right]^{\frac{1}{2}}\mathrm{d}x=(2m\mp1)\pi\mathrm{i}. \tag{6.17}$$

□

The following remark is in order.

If $F_{1,2}(\phi)$ diverged to infinity faster than ϕ_2 then every integral of $f(F_{1,2}^{\pm})$ in (6.13) would contain an E-dependent infinite series not reducing, of course, to simple values $\pm\mathrm{i}\pi$: i.e., relation (6.16) as well as (6.17) could no longer be valid.

However, it is shown in appendix D that *any* integer power of the divergency of $F_{1,2}(\phi)$ to infinity is allowed by the shape invariance condition (6.12). Therefore, the bound on this power introduced to theorem 2 by its assumption (2), is essential. It is still, however, worth noting that the bound in the last assumption can be substituted equivalently by another property of the superpotential $\phi(x,\lambda,a)$ if the latter is shape invariant (i.e. it satisfies (6.12) if this condition

is expressed in terms of this superpotential). The discussed assumption is equivalent to the demand that the *shifted* superpotential $\phi_1 = \phi(x, \lambda, a_1),\ a_1 = f(a)$, when expressed as a function of $\phi = \phi(x, \lambda, a)$ (i.e. $\phi_1 = \tilde{\phi}(\phi, \lambda, a)$ diverges linearly with the latter when $\phi \to \infty$, whilst the corresponding proportionality coefficient approaches unity when $\lambda \to \infty$. All the 11 potentials (6.3) satisfy this demand. It is shown in appendix D that the shape invariant conditon (6.12) can be then fulfiled only if $F_2(\phi, \lambda, a)$ does not diverge faster than ϕ^2.

7. Discussion and conclusions

In this paper we have shown that the early success of the exact JWKB and SJWKB formulae was related to the simple singularity and turning point structures of potentials. In particular, this was facilitated by having no more than two (occasionally four) turning points and no more than one second-order pole in the basic period strip. In the opposite case, an unavoidable proliferation of additional sectors in the basic period strip prevents the periodicity properties of the corresponding quantization conditions (2.7) to be used to reduce the conditions to the pure JWKB ones. The possible relaxation of these conditions has been described in section 5 and the corresponding examples were given in an earlier paper by one of us [1].

The above simplicity conditions reduced effectively a number of exactly JWKB-quantized potentials to only 11. *All* of them have long been known. However, due to the investigations above they have been given the status of being rather exceptional.

Furthermore, we have shown that the sufficient condition for a potential to be exactly JWKB(SJWKB)-quantized is the shape invariance of the latter supported by the holomorphicity of the functions $F_{1,2}(x, E, \lambda)$ on the Riemann surface of the corresponding superpotential and their proper asymptotic behaviour on the surface. If the latter properties are satisfied the energy levels are quantized exactly and simultaneously by the JWKB formulae of both types: conventional and SUSY. The two theorems of section 6 give necessary and sufficient conditions for this to happen. Both the theorems show the close and direct relation between the exactness of the JWKB and SJWKB formulae which follows on the one hand from the shape invariance of the potentials and on the other hand from the singularity structures of the latter on the x- and ϕ-planes. In fact, we can conclude that the results of section 3 are nothing but the solutions to the shape invariance symmetry condition (6.12) expressed in terms of the allowed numbers of turning points and poles in the basic period strips.

We must note also that the results obtained by Inomata *et al* [26] for the form of the SUSY JWKB formulae for broken SUSY potentials do not contradict ours which always choose the form of Comtet *et al* [7] since the latter concern their exact, unapproximated forms which are the subject of Inomata *et al*. However, despite having the same form, formula (6.2) gives *different* results for energy levels depending on whether the supersymmetry is exact or broken: in the latter case reproducing effectively the result of Inomata *et al*.

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Appendix A

Here we shall construct the Weierstrass product representation for the following holomorphic 2πi-periodic function:

$$q(x, E, \lambda) = \sum_{n=l}^{k} q_n(E, \lambda)\mathrm{e}^{nx} \tag{A.1}$$

with $k > l$ and *even* $k-l$ and having only simple zeros.

Let $y_1, \ldots, y_{k-l}$ be the (simple) zeros of the polynomial $q(\log y, E, \lambda)/y^l$, then $x_{p,n} = \log y_p + 2\pi \mathrm{i}n$, $p = 1, \ldots, k-l, n = 0, \pm 1, \pm 2, \ldots$, exhaust all zeros of $q(x, E, \lambda)$. We shall show that $q(x, E, \lambda)$ can be represented by

$$\begin{aligned} q(x, E, \lambda) = C\mathrm{e}^{\frac{k+l}{2}x} x \left[1 - \frac{x}{x_{2,0}}\right] \cdots \left[1 - \frac{x}{x_{k-l,0}}\right] \\ \times \prod_{p=1}^{k-l} \prod_{n \geqslant 1} \left[1 - \frac{x}{\log y_p + 2\pi \mathrm{i}n}\right] \left[1 - \frac{x}{\log y_p - 2\pi \mathrm{i}n}\right] \end{aligned} \tag{A.2}$$

where $C = q(x, E, \lambda)/x|_{x=0}$ or $C = q(0, E, \lambda)$ if $x = 0$ is not a root of $q(x, E, \lambda)$.

The above formula follows from the observation that $Q(x, E, \lambda) = q(x, E, \lambda) \exp[(-k/2 - l/2)x]$ is also $2\pi i$-periodic and holomorphic with the same roots as $q(x, E, \lambda)$ and from another observation that the convergence-producing exponentials in the rhs of (A.2) can be shifted to the front of the product if the latter is taken in the form shown above. Therefore, the Weierstrass product representation of $Q(x, E, \lambda)$ can be given in the following form:

$$\begin{aligned} Q(x, E, \lambda) = C\mathrm{e}^{\alpha x} x \left[1 - \frac{x}{x_{2,0}}\right] \cdots \left[1 - \frac{x}{x_{k-l,0}}\right] \\ \times \prod_{p=1}^{k-l} \prod_{n \geqslant 1} \left[1 - \frac{x}{\log y_p + 2\pi \mathrm{i}n}\right] \left[1 - \frac{x}{\log y_p - 2\pi \mathrm{i}n}\right] \end{aligned} \tag{A.3}$$

where α is an integer by periodicity of Q. In general, α should depend analytically on the coefficients q_n of (A.1) but being an integer it is a *constant* function of the latter. Therefore, to establish its value we can choose some appropriate point in the space of q_n: namely, the one for which $Q(x, E, \lambda)$, as defined by (A.3), becomes an even function of x under the reflection $x \to -x$. To achieve this goal it is enough to continue q_n to the point where $q_n = q_{k+l-n}$, $n = l, l+1, \ldots, (k+l)/2$. Then, the operation $x \to -x$ does not change in (A.3) the product itself (the distribution of roots are then invariant under the operation) but changes $\mathrm{e}^{\alpha x}$ into $\mathrm{e}^{-\alpha x}$. Therefore, $\alpha = 0$.

As an example, consider $q(x, E, \lambda)$ given by (3.2) for which its distribution of roots is shown in figure 2. We have for it

$$\begin{aligned} \alpha\mathrm{e}^{2x} - 2\beta\mathrm{e}^{x} + \gamma = (\alpha - 2\beta + \gamma)\mathrm{e}^{x} \left[1 - \frac{x}{x_+}\right] \left[1 - \frac{x}{x_-}\right] \\ \times \prod_{p=\pm} \prod_{n \geqslant 1} \left[1 - \frac{x}{x_p + 2\pi \mathrm{i}n}\right] \left[1 - \frac{x}{x_p - 2\pi \mathrm{i}n}\right]. \end{aligned} \tag{A.4}$$

We want to calculate with the help of (A.4) a change of phase of $q(x, E, \lambda)$ when transporting it from point x_0 of the line $\mathrm{Im}\, x = \pi$ to the point $x_0 - 2\pi\mathrm{i}$ of the line $\mathrm{Im}\, x = -\pi$. We note that as follows from (A.4), the roots of $q(x, E, \lambda)$ lying at large distances from the points considered have almost no contribution to the values of $q(x, E, \lambda)$ in the considered strip (their product in (A.4) is close to 1). Therefore, we can take a sufficiently large but finite

number of roots around the considered points to perform the calculations needed (eventually we can take the limit of the infinite number of roots).

Starting from point x_0 we can consider n pairs of roots lying above the line $\operatorname{Im} x = \pi$ (n is large) and n pairs of roots lying below the line. The arguments of $x_0 - x_k$ we take to be positive for x_k lying below the line $\operatorname{Im} x = \pi$ and negative in the opposite case. It is clear that the net result of summing the corresponding arguments of the product in (A.4) is zero. There is, however, still a non-zero contribution to the argument of $q(x_0, E, \lambda)$ coming from the factor e^x of (A.4). It amounts, of course, to π and this is the total argument of $q(x_0, E, \lambda)$.

At the points $x_0 - \pi i$ our calculations are similar. Keeping the *same* set of roots as chosen previously we see that to the total phase of the product at $x_0 - \pi i$ contribute only the two most distant pairs of roots lying *above* the line $\operatorname{Im} x = \pi$, so according to our convention this contribution amounts to $4(-\pi/2) = -2\pi$ (in the limit of the root number going to infinity). Together with the argument $-\pi$ provided by the factor e^x we get the argument of $q(x_0 - \pi i, E, \lambda)$ to be equal to -3π. Therefore, the total change of the argument of $q(x, E, \lambda)$ between the lines considered is equal to -4π.

Appendix B

We shall collect here all the superpotentials ϕ_k corresponding to the potentials V_k, $k = 1, \ldots, 8$: i.e. to the first eight listed in section 4 and to the last two discussed in the latter part of that section (formula (3.18)). We shall also collect all their broken partners with the corresponding method of symmetry breaking described by points 2^0–4^0 in section 4. The corresponding functions $F_1(\phi)$ and $F_2(\phi)$ are also given together with the coefficients a_k, b_k, c_k, $k = 1, 2$, of their asymptotic expansions when $\phi \to \infty$.

1^0

$$\phi_1(x, \lambda) = |\alpha| e^x - \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}$$

$$F_1(\phi_1) = F_2(\phi_1) = \phi_1 + \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}$$

$$b_1 = 1 \qquad \epsilon_0 = -\left(\frac{\beta}{|\alpha|} - \frac{1}{2\lambda}\right)^2.$$

4^0

$$\phi_1(x, \lambda) = -|\alpha| e^x + \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}$$

$$F_1(\phi_1) = F_2(\phi_1) = \phi_1 - \frac{\beta}{|\alpha|} + \frac{1}{2\lambda}$$

$$b_1 = 1 \qquad \epsilon_0 = -\left(\frac{\beta}{|\alpha|} + \frac{1}{2\lambda}\right)^2.$$

1^0

$$\phi_2(x, \lambda) = -\frac{|2l+1|+1}{2\lambda x} + \frac{\lambda\alpha}{|2l+1|+1}$$

$$F_1(\phi_2) = \lambda(2|2l+1|+1)\frac{(\phi_2 - \frac{\alpha\lambda}{|2l+1|+1})^2}{(|2l+1|+1)^2}$$

$$F_2(\phi_2) = 2\lambda \frac{(\phi_2 - \frac{\alpha\lambda}{|2l+1|+1})^2}{|2l+1|+1}$$
$$c_1 = \lambda \frac{2|2l+1|+1}{(|2l+1|+1)^2} \qquad c_2 = \frac{2\lambda}{|2l+1|+1}$$
$$\epsilon_0 = -\frac{(\lambda\alpha)^2}{(|2l+1|+1)^2}$$
$$\beta = \frac{l(l+1)}{\lambda^2} > 0.$$

4^0

$$\phi_2(x,\lambda) = \frac{|2l+1|-1}{2\lambda x} - \frac{\lambda\alpha}{|2l+1|-1}$$
$$F_1(\phi_2) = -\lambda(2|2l+1|-1)\frac{(\phi_2 + \frac{\alpha\lambda}{|2l+1|-1})^2}{(|2l+1|-1)^2}$$
$$F_2(\phi_2) = -2\lambda \frac{(\phi_2 + \frac{\alpha\lambda}{|2l+1|-1})^2}{|2l+1|-1} \qquad c_1 = -\lambda \frac{2|2l+1|-1}{(|2l+1|-1)^2} \qquad c_2 = -\frac{2\lambda}{|2l+1|-1}$$
$$\epsilon_0 = -\frac{(\lambda\alpha)^2}{(|2l+1|-1)^2}$$
$$\beta = \frac{l(l+1)}{\lambda^2} > 0.$$

1^0

$$\phi_3(x,\lambda) = |\alpha|x - \frac{|2l+1|+1}{2\lambda x}$$
$$F_1(\phi_3) = \lambda(2|2l+1|+1)\left[\phi_3^2 + 2|\alpha|\frac{|2l+1|+1}{\lambda}\right]^{\frac{1}{2}} \frac{\phi_3 + (\phi_3^2 + 2|\alpha|\frac{|2l+1|+1}{\lambda})^{\frac{1}{2}}}{2(|2l+1|+1)^2} + \frac{|\alpha|}{2|2l+1|+2}$$
$$F_2(\phi_3) = \lambda\left[\phi_3^2 + 2|\alpha|\frac{|2l+1|+1}{\lambda}\right]^{\frac{1}{2}} \frac{\phi_3 + (\phi_3^2 + 2|\alpha|\frac{|2l+1|+1}{\lambda})^{\frac{1}{2}}}{|2l+1|+1}$$
$$c_1^{(1)} = \lambda \frac{2|2l+1|+1}{(|2l+1|+1)^2} \qquad c_2^{(1)} = \frac{2\lambda}{|2l+1|+1}$$
$$c_1(2) = c_2(2) = 0$$
$$\epsilon_0 = (|2l+1|+2)\frac{|\alpha|}{\lambda} \qquad \beta = \frac{l(l+1)}{\lambda^2} > 0.$$

2^0: we get this case from 1^0, substituting $|2l+1|$ by $-|2l+1|$;
3^0: we get this case from 1^0 by the substitution $|\alpha| \to -|\alpha|$;
4^0: we get this case from 1^0, substituting $|2l+1|$ by $-|2l+1|$ and $|\alpha|$ by $-|\alpha|$.

1^0

$$\phi_4 = \frac{|2l+1|-1}{4\lambda}\tanh\frac{x}{2} + \frac{4\lambda\alpha}{|2l+1|-1}$$
$$F_1(\phi_4) = -\lambda(2|2l+1|-1)\left(\frac{\phi_4 - \frac{4\lambda\alpha}{|2l+1|-1}}{|2l+1|-1}\right)^2 + \frac{2|2l+1|-1}{16\lambda}$$

$$F_2(\phi_4) = -2\lambda\frac{(\phi_4 - \frac{4\lambda\alpha}{|2l+1|-1})^2}{|2l+1|-1} + \frac{|2l+1|-1}{8\lambda}$$

$$c_1 = -\lambda\frac{2|2l+1|-1}{(|2l+1|-1)^2} \qquad c_2 = -\frac{2\lambda}{|2l+1|-1}$$

$$\epsilon_0 = -\left[\frac{|2l+1|-1}{2\lambda} - \frac{2\lambda\alpha}{|2l+1|-1}\right]^2$$

$$\alpha + \beta = \frac{l(l+1)}{(2\lambda)^2} > 0.$$

4^0: we get this case from 1^0 by the substitution $|2l+1| \to -|2l+1|$.

1^0

$$\phi_5 = -\frac{|2l+1|+1}{4\lambda}\coth\frac{x}{2} - \frac{4\lambda\alpha}{|2l+1|+1}$$

$$F_1(\phi_5) = \lambda(2|2l+1|+1)\left(\frac{\phi_5 + (\frac{4\lambda\alpha}{|2l+1|+1})^2}{|2l+1|+1}\right)^2 - \frac{2|2l+1|+1}{16\lambda}$$

$$F_2(\phi_5) = +2\lambda\frac{(\phi_5 + \frac{4\lambda\alpha}{|2l+1|+1})^2}{|2l+1|+1} - \frac{|2l+1|+1}{8\lambda}$$

$$c_1 = \lambda\frac{2|2l+1|+1}{(|2l+1|+1)^2} \qquad c_2 = \frac{2\lambda}{|2l+1|+1}$$

$$a_1^\infty = a_2^\infty = |\alpha| \qquad \epsilon_0 = -\left[-\frac{|2l+1|+1}{2\lambda} + \frac{2\lambda\alpha}{|2l+1|+1}\right]^2$$

$$\alpha + \beta = \frac{l(l+1)}{(2\lambda)^2} > 0.$$

4^0: we get this case from 1^0 by the substitution $|2l+1| \to -|2l+1|$.

1^0

$$\phi_6 = \frac{|2l+1|-1}{4\lambda}\tanh\frac{x}{2} - \frac{|2l'+1|+1}{4\lambda}\coth\frac{x}{2}$$

$$F_1(\phi_6) = -\frac{\lambda}{2}(2|2l+1|-1)\left[\phi_6^2 + \frac{(2|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}\right]^{\frac{1}{2}}$$
$$\times\frac{\phi_6 + [\phi_6^2 + \frac{(2|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}]^{\frac{1}{2}}}{(|2l+1|-1)^2}$$
$$-\frac{\lambda}{2}(2|2l'+1|-1)\left[\phi_6^2 + \frac{(2|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}\right]^{\frac{1}{2}}$$
$$\cdot\frac{\phi_6 - [\phi_6^2 + \frac{(|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}]^{\frac{1}{2}}}{(|2l'+1|+1)^2}$$

$$F_2(\phi_6) = -\lambda\left[\phi_6^2 + \frac{(|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}\right]^{\frac{1}{2}}\frac{\phi_6 + [\phi_6^2 + \frac{(|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}]^{\frac{1}{2}}}{(|2l+1|-1)^2}$$
$$-\lambda\left[\phi_6^2 + \frac{(|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}\right]^{\frac{1}{2}}\frac{\phi_6 - [\phi_6^2 + \frac{(|2l+1|-1)(|2l'+1|+1)}{(2\lambda)^2}]^{\frac{1}{2}}}{(|2l'+1|+1)^2}$$

$$c_1^{(1)} = \lambda\frac{2|2l'+1|+1}{(|2l'+1|+1)^2} \qquad c_1^{(2)} = \lambda\frac{2|2l+1|-1}{(|2l+1|-1)^2}$$

$$c_2^{(1)} = \frac{2\lambda}{|2l'+1|+1} \qquad c_2^{(2)} = -\frac{2\lambda}{|2l+1|-1}$$
$$\epsilon_0 = \frac{(l-l'-1)^2}{(2\lambda)^2} \qquad \alpha = \frac{l(l+1)}{(2\lambda)^2} > 0 \qquad \beta = \frac{l'(l'+1)}{(2\lambda)^2} > 0$$
$$|2l+1| - |2l'+1| > 2.$$

2^0: we get this case from 1^0 taking l, l', satisfying $|2l+1|-|2l'+1| < 2$ or substituting $|2l+1|$ by $-|2l+1|$;

3^0: we get this case from 1^0 substituting $|2l'+1|$ by $-|2l'+1|$ and next taking l, l', satisfying $|2l'+1| \pm |2l+1| > 2$;

4^0: we get this case from 1^0 substituting $|2l'+1|$ by $-|2l'+1|$ and next taking l, l', satisfying $|2l'+1| \pm |2l+1| < 2$.

1^0

$$\phi_7 = -\frac{|2l+1|-1}{4\lambda}\tan\frac{x}{2}$$
$$F_1(\phi_7) = -\lambda(2|2l+1|-1)\frac{\phi_7^2}{(|2l+1|-1)^2} - \frac{2|2l+1|-1}{16\lambda}$$
$$F_2(\phi_7) = -2\lambda\frac{\phi_7^2}{|2l+1|-1} - \frac{|2l+1|-1}{8\lambda}$$
$$c_1 = -\lambda\frac{2|2l+1|-1}{(|2l+1|-1)^2} \qquad c_2 = -\frac{2\lambda}{|2l+1|-1}$$
$$\epsilon_0 = \left(\frac{|2l+1|-1}{4\lambda}\right)^2$$
$$\alpha = \frac{l(l+1)}{(2\lambda)^2} > 0.$$

4^0: we get this case from 1^0 substituting $|2l+1|$ by $-|2l+1|$.

1^0

$$\phi_8 = \frac{|2l+1|+1}{4\lambda}\tan\frac{x}{2} - \frac{|2l'+1|+1}{4\lambda}\cot\frac{x}{2}$$
$$F_1(\phi_8) = \frac{\lambda}{2}(2|2l+1|+1)\left[\phi_8^2 + \frac{(|2l'+1|+1)(|2l+1|+1)}{(4\lambda)^2}\right]^{\frac{1}{2}}$$
$$\times\frac{\phi_8 + [\phi_8^2 + \frac{(|2l'+1|+1)(|2l+1|+1)}{(4\lambda)^2}]^{\frac{1}{2}}}{(|2l+1|+1)^2} - \frac{\lambda}{2}(2|2l'+1|+1)$$
$$\times\left[\phi_8^2 + \frac{(|2l'+1|+1)(|2l+1|+1)}{4\lambda^2}\right]^{\frac{1}{2}}$$
$$\times\frac{\phi_8 - [\phi_8^2 + \frac{(|2l'+1|-1)(|2l+1|+1)}{(4\lambda)^2}]^{\frac{1}{2}}}{(|2l'+1|+1)^2} + \frac{(|2l+1|-|2l'+1|)^2}{16\lambda(|2l+1|+1)(|2l'+1|+1)}$$
$$F_2(\phi_8) = \lambda\left[\phi_8^2 + \frac{(|2l'+1|-1)(|2l+1|+1)}{(4\lambda)^2}\right]^{\frac{1}{2}}\frac{\phi_8 + [\phi_8^2 + \frac{(|2l'+1|-1)(|2l+1|+1)}{(4\lambda)^2}]^{\frac{1}{2}}}{|2l+1|+1}$$
$$-\lambda\left[\phi_8^2 + \frac{(2|l'+1|-1)(|2l+1|+1)}{(4\lambda)^2}\right]^{\frac{1}{2}}\frac{\phi_8 - [\phi_8^2 + \frac{(|2l'+1|-1)(|2l+1|+1)}{(4\lambda)^2}]^{\frac{1}{2}}}{(|2l'+1|-1)^2}$$
$$c_1^{(1)} = \lambda\frac{2|2l+1|+1}{(|2l+1|+1)^2} \qquad c_1^{(2)} = -\lambda\frac{2|2l'+1|+1}{(|2l'+1|+1)^2}$$

$$c_2^{(1)} = \frac{2\lambda}{|2l+1|+1} \qquad c_2^{(2)} = -\frac{2\lambda}{|2l'+1|+1}$$

$$\epsilon_0 = \frac{1}{(2\lambda)^2}\left(l(l+1)+l'(l'+1)+\frac{1}{2}(|2l+1|+|2l'+1|+2) + \frac{1}{2}(|2l+1|+1)(|2l'+1|+1)\right)$$

$$\alpha = \frac{l(l+1)}{(2\lambda)^2} > 0 \qquad \beta = \frac{l'(l'+1)}{(2\lambda)^2} > 0.$$

2^0: we get this case from 1^0, substituting $|2l+1|$ by $-|2l+1|$;
3^0: we get this case from 1^0, substituting $|2l'+1|$ by $-|2l'+1|$;
4^0: we get this case from 1^0, substituting $|2l+1|$ by $-|2l+1|$ as well as $|2l'+1|$ by $-|2l'+1|$.

1^0

$$\phi_9 = a\tan x + \frac{b}{\cos x}$$

$$F_1(\phi_9) = \frac{1}{(a^2-b^2)^2}\left(b\phi - a\sqrt{\phi^2+a^2-b^2}\right) \left[a'\left(b\phi - a\sqrt{\phi^2+a^2-b^2}\right) - b\left(a\phi - b\sqrt{\phi^2+a^2-b^2}\right)\right]$$

$$F_2(\phi_9) = \frac{1}{(a^2-b^2)^2}\left(b\phi - a\sqrt{\phi^2+a^2-b^2}\right) \left[a\left(b\phi - a\sqrt{\phi^2+a^2-b^2}\right) - b\left(a\phi - b\sqrt{\phi^2+a^2-b^2}\right)\right]$$

$$c_1^{(1)} = \frac{a'+b}{(a+b)^2} \qquad c_1^{(2)} = \frac{a'-b}{(a-b)^2}$$

$$c_2^{(1)} = \frac{1}{a+b} \qquad c_2^{(2)} = \frac{1}{a-b}$$

$$\alpha+\beta = \frac{l(l+1)}{\lambda^2} > 0 \qquad \alpha-\beta = \frac{l'(l'+1)}{\lambda^2} > 0$$

$$a = \frac{|2l+1|+|2l'+1|}{4\lambda} + \frac{1}{2\lambda} \qquad a' = a - \frac{1}{4\lambda}$$

$$b = \frac{|2l+1|-|2l'+1|}{4\lambda}$$

2^0: we get this case substituting in 1^0, $|2l+1|$ by $-|2l+1|$;
3^0: we get this case substituting in 1^0, $|2l'+1|$ by $-|2l'+1|$;
4^0: we get this case substituting in 1^0, $|2l+1|$ by $-|2l+1|$ and $|2l'+1|$ by $-|2l'+1|$.

1^0

$$\phi_{10} = a\tanh x + \frac{b}{\cosh x}$$

$$F_1(\phi_{10}) = \frac{1}{4\lambda}\frac{(\phi_{10}^2-a^2)^2 + 4\lambda(\phi_{10}^2-a^2)\left(a(\phi_{10}^2-a^2-b^2) - \mathrm{i}b\phi_{10}\sqrt{\phi_{10}^2-a^2-b^2}\right)}{(b^2-a^2)(\phi_{10}^2-a^2) + 2a^2b^2 + 2\mathrm{i}ab\phi_{10}\sqrt{\phi_{10}^2-a^2-b^2}}$$

$$F_2(\phi_{10}) = (\phi_{10}^2-a^2)\frac{a(\phi_{10}^2-a^2-b^2) - \mathrm{i}b\phi_{10}\sqrt{\phi_{10}^2-a^2-b^2}}{(b^2-a^2)(\phi_{10}^2-a^2) + 2a^2b^2 + 2\mathrm{i}ab\phi_{10}\sqrt{\phi_{10}^2-a^2-b^2}}$$

$$c_1^{(1)} = -\frac{1}{a-\mathrm{i}b} - \frac{1}{4\lambda}\frac{1}{(a-\mathrm{i}b)^2} \qquad c_1^{(2)} = -\frac{1}{a+\mathrm{i}b} - \frac{1}{4\lambda}\frac{1}{(a+\mathrm{i}b)^2}$$
$$c_2^{(1)} = -\frac{1}{a-\mathrm{i}b} \qquad c_2^{(2)} = -\frac{1}{a+\mathrm{i}b}$$
$$\alpha = b^2 - \frac{l(l+1)}{\lambda^2} \qquad \beta = |b|(|2l+1|+2) \qquad a = \frac{|2l+1|+1}{2\lambda} > 0$$
$$b > 0 \qquad l(l+1) > 0.$$

4^0: we get this case substituting in 1^0, $|2l+1|$ by $-|2l+1|$ or allowing l to vary in 1^0 in the segment $-1 < l < 0$ (the allowed b is then negative in both cases).

1^0

$$\phi_{11}(x,\lambda) = |\alpha|x \qquad \phi'_{11}(x,\lambda) = |\alpha|$$
$$F_1(\phi,\lambda) = F_2(\phi,\lambda) \equiv |\alpha|$$
$$\epsilon_0 = \frac{|\alpha|}{\lambda}.$$

4^0: we get this case substituting in 1^0, $|\alpha|$ by $-|\alpha|$.

Appendix C

We demonstrate here the particularities of obtaining formula (6.10) for the potential $V_9(x,\lambda) \equiv V_{9,-}(x,\lambda)$.

There are three possibilities of breaking spontaneously the supersymmetry for this case. We shall choose only one of them corresponding to the first of the three cases listed in appendix B.

Consider, therefore, relation (6.14) using the superpotentials $\phi_9(x,\lambda)$ given above. First consider case 1^0 of the exact supersymmetry. The corresponding Riemann surface R_{ϕ_9} is depicted in figure 10. This is a two-sheeted surface with the branch points at $\phi_9 = \pm\mathrm{i}(a^2-b^2)^{\frac{1}{2}}$. The latter are the unique singularities of the integrand of the following integral:

$$\oint_{K_{\phi_9}} \left[\sqrt{\phi_9^2 - \frac{1}{\lambda}F_1(\phi_9) - \tilde{E}} - \sqrt{\phi_9^2 - \tilde{E}}\right] \frac{\mathrm{d}\phi_9}{F_2(\phi_9)} \tag{C.1}$$

(since the roots of F_2 at $\phi_9 = \pm\mathrm{i}a$ are also the roots of F_1).

R_{ϕ_9} is, clearly, a map of the basic period strip $-\pi \leqslant x \leqslant \pi$ of the x-plane (see figure 9), so that the four turning points of $q_{9,-}(x,\lambda,E)$ from this strip are mapped pairwise into R_{ϕ_9}: the two from segment $(-\pi/2, \pi/2)$ into sheet (a) of figure 10 and the other two into the second one. It is also easy to note that in quantization formulae (6.1) and (6.2) the contour K in figure 9 can be substituted by contour K', surrounding the next two turning points, the latter contour being related to the joined symmetry operations: $x \to x+\pi$ and $x \to -x$. The contours are mapped into R_{ϕ_9} as $K_{1,\phi}$ and $K_{2,\phi}$, respectively, the latter surrounding the respective pairs of the turning points pictured on R_{ϕ_9} (see figure 10). Therefore, for the quantization formulae (6.1) and (6.2) we can write

$$-\lambda \oint_{K_1} \sqrt{\phi_9^2 - \frac{1}{\lambda}\phi'_9 + \frac{\delta}{\lambda^2} - \tilde{E}}\,\mathrm{d}x = -\frac{\lambda}{2}\left[\oint_{K_1} + \oint_{K_2}\right]\sqrt{\phi_9^2 - \frac{1}{\lambda}\phi'_9 + \frac{\delta}{\lambda^2} - \tilde{E}}\,\mathrm{d}x$$
$$= -\lambda \oint_{K_1} \sqrt{\phi_9^2 - \tilde{E}}\,\mathrm{d}x - \frac{\lambda}{2}\left[\oint_{K_{1,\phi}} + \oint_{K_{2,\phi}}\right]$$

$$\times\left[\sqrt{\phi_9^2-\frac{1}{\lambda}F_1(\phi_9)-\tilde{E}}-\sqrt{\phi_9^2-\tilde{E}}\right]\frac{\mathrm{d}\phi_9}{F_2(\phi_9)}$$
$$=-\lambda\oint_{K_1}\sqrt{\phi_9^2-\tilde{E}}\,\mathrm{d}x-\frac{\lambda}{2}$$
$$\times\left[\oint_{K_{\infty_1,\phi}}+\oint_{K_{\infty_2,\phi}}\right]\left[\sqrt{\phi_9^2-\frac{1}{\lambda}F_1(\phi_9)-\tilde{E}}-\sqrt{\phi_9^2-\tilde{E}}\right]\frac{\mathrm{d}\phi_9}{F_2(\phi_9)} \tag{C.2}$$

where $K_{\infty_1,\phi}$ and $K_{\infty_2,\phi}$ are the contours obtained by obvious deformations of the contours $K_{1,\phi}$ and $K_{2,\phi}$, which contain all the singularities of $F_{1,2}(\phi_9)$. Making use of the explicit forms of $F_{1,2}(\phi_9)$ as given in section 4, we can calculate the last integral in (C.2) obtaining for it the value $+\mathrm{i}\pi$. Together with (6.2) this gives result (6.1).

Consider now the broken case 2^0 of the superpotential ϕ_9. The corresponding basic period strip of $q_{9,-}(x,\lambda,E)$ and the quantization contours K_1 and K_2 transform into R_{ϕ_9}, as is shown in figure 11. Once again, we can write the sequence analogous to (C.2) by deforming the contours $K_{1,\phi}$ and $K_{2,\phi}$ of figure 11 into $K_{\infty_1,\phi}$ and $K_{\infty_2,\phi}$, respectively, to perform the final integration obtaining *again* $+\mathrm{i}\pi$ and consequently the exact formula (6.1). Of course, the starting value of m can now be zero.

Appendix D

We shall show here that the shape invariance condition (6.12) does not prevent, in some obvious way, $F_{1,2}(\phi)$ to diverge with any power of ϕ when $\phi\to\infty$. But if we limit the corresponding divergence of the 'shifted' superpotential $\phi_1=\phi(x,\lambda,a_1)$ $a_1=f(a)$ to the linear one then the divergence of $F_2(\phi)$ cannot be faster than ϕ^2.

To this end, let us rewrite (6.12) in terms of superpotentials. We get

$$\begin{aligned}&\phi^2(x,\lambda,a)+\frac{1}{\lambda}\phi'(x,\lambda,a)=\phi^2(x,\lambda,a_1)-\frac{1}{\lambda}\phi'(x,\lambda,a_1)+R(a_1)\\&a_1=f(a).\end{aligned} \tag{D.1}$$

Equation (D.1) is satisfied when x is taken to be the same on both the sides.

Putting $\phi\equiv\phi(x,\lambda,a)$ and $\phi_1\equiv\phi(x,\lambda,a_1)$ and inverting both the latter identities with respect to x, we can write

$$x(\phi,\lambda,a)=x(\phi_1,\lambda,a_1). \tag{D.2}$$

Next, taking into account that in (D.2) $a_1=f(a)$ we can solve the latter with respect to ϕ_1 to get

$$\phi_1=\tilde{\phi}(\phi,\lambda,a)\equiv\phi(x(\phi,\lambda,a),\lambda,f(a)). \tag{D.3}$$

Introducing further the latter function, as well as the function $F_2(\phi,a)$ ($\equiv\phi'(x(\phi,a),a)$) to (D.1), we obtain

$$\phi^2+\frac{1}{\lambda}F_2(\phi,\lambda,a)=\tilde{\phi}^2(\phi,\lambda,a)-\frac{1}{\lambda}\tilde{\phi}'_\phi(\phi,\lambda,a)F_2(\phi,\lambda,a)+R(a_1). \tag{D.4}$$

Consider now (D.4) on some sheet of the ϕ-Riemann surface upon which this equation is defined. Then assuming that for ϕ large enough all the terms in (D.4) behave holomorphicly we can easily see that (D.4) is satisfied asymptotically for $\phi\to\infty$ if $\tilde{\phi}_1$ grows linearly with ϕ, whilst $F_2(\phi,\lambda,a)$ can grow with *any* finite power of ϕ.

However, if this growth is faster than ϕ^2 then (D.4) can be satisfied only if $\tilde{\phi}\sim-\phi$ with $\phi\to\infty$. On the other hand, if $\tilde{\phi}\sim(1+\frac{c}{\lambda})\phi$ in the last limit then from (D.4) it follows easily that in such a case $F_2(\phi,\lambda,a)\sim c_2\phi^2$.

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